Twistor geometry

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Contents

1 Introduction

Our aim is to give a view of the geometrical relationship between Minkowski space, the space-time of special relativity, and projective twistor space, which is $\mathbb{CP}^3$. We also give a hint as to how the *Penrose transform* takes functions on twistor space to solutions of field equations in space-time. We refer to [2] for details.

But we warm up by looking at a piece of classical geometry. Given a circle centre O and radius r in $\mathbb{R}^2$, *inversion* in this circle is the mapping

$$t : \mathbb{R}^2 \setminus \{O\} \to \mathbb{R}^2 \setminus \{O\}$$

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defined by $t(A) = A'$, where A' lies on the straight line through O and A, and on the same side of O as A, and $OA.OA' = r^2$. This mapping has many lovely properties but is not defined at O. We can fix this by adding to $\mathbb{R}^2$ a single point at infinity, as follows.

Place a sphere on a plane, so that the point of contact is the south pole. Let P be a point on the sphere, other than the north pole N. Draw a straight line from N through P, and suppose that it intersects the plane at Q. *Stereographic projection* is the mapping

$$\sigma : S^2 \setminus \{N\} \longrightarrow \mathbb{R}^2 \tag{1}$$

$$\sigma(P) \longmapsto Q \tag{2}$$

It is a homeomorphism. The sphere has exactly one extra point, N, which is "at infinity" in $\mathbb{R}^2$. S^2 is called the one-point *compactification* of $\mathbb{R}^2$.

Circles and straight lines in $\mathbb{R}^2$ are mapped by σ^{-1} to circles on the sphere, and, in particular, inversions in circles and reflections in straight lines in $\mathbb{R}^2$ are well-defined mappings from S^2 to itself. The group of such mappings contains the Euclidean group as a subgroup (because the Euclidean group can be generated by reflections).

In the next section we will see how to define inversions in space-time, and how to compactify space-time so that these inversions are well-defined.

2 Minkowski space

A point of space-time is an *event* (which is a point in space and an instant in time). An observer S in space-time needs four coordinates (t, x, y, z) to describe the events she sees. Suppose another observer S', moving with respect to S, uses the coordinates (t', x', y', z'). We choose the x', y', and z' axes to coincide with the x, y, and z axes when $t = 0$, and we also choose $t' = 0$ then too. Subsequently the origin of S' moves at constant speed v along the x axis of S, keeping the y and y' axes parallel. Note that we also choose S to be an *inertial* frame of reference, which means that it is not accelerating. Then S' is also inertial.

Motivated by a desire to unify Newtonian mechanics with Maxwell's theory of electromagnetism, Einstein adopted the following two physical principles.

(a) All inertial frames of reference are equivalent for all physical experiments.

(b) Light has the same velocity c in all inertial frames of reference.

Einstein discovered that as a direct consequence of these two principles, the transformation between S and S' must be given by the *standard Lorentz transformation*

$$t' = \gamma(t - vx/c^2) \qquad x' = \gamma(x - vt) \qquad y' = y \qquad z' = z, \tag{3}$$

where

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}.$$

Minkowski space $\mathbb{M}$, the space-time of special relativity, is a four dimensional real vector space. It has a metric

$$dx^a dx^b g_{ab} = (dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2 \tag{4}$$

where $x^0 = ct, x^1 = x, x^2 = y, x^3 = z$. This metric can be positive, zero, or negative, and so at each event

$$x^a = (x^0, x^1, x^2, x^3)$$

in $\mathbb{M}$ there is a *null cone*, distinguishing these three cases.

For a mathematician, special relativity is the geometry of $\mathbb{M}$ under the transformations which are isometries of the metric (4). These transformations form the *Lorentz group* $O(1,3)$, which we can enlarge by including translations to form the *Poincaré group*.

Exercise 1 *Show that the transformation (3) is in the Lorentz group.*

However, Maxwell's equations for electromagnetism are in fact invariant under an even larger group: the group of *conformal transformations*. This group consists of the Poincaré group together with dilations and inversions.

An inversion in Minkowski space has the formula

$$x^a \longmapsto \frac{x^a}{\Delta}, \tag{5}$$

where

$$\Delta = x^a x^b g_{ab}.$$

However, this transformation (5) is not well defined on all of $\mathbb{M}$, because it is singular on the null cone $\Delta = 0$. To overcome this problem we need to *compactify* Minkowski space.

Consider $\mathbb{R}^6$ with coordinates (T, V, W, X, Y, Z) and the metric

$$dT^2 + dV^2 - dW^2 - dX^2 - dY^2 - dZ^2. \tag{6}$$

We use the mapping $\mathbb{M} \longrightarrow \mathbb{R}^6$ given by

$$x^a \longmapsto (x^0, \frac{1}{2}(1-\Delta), -\frac{1}{2}(1+\Delta), x^1, x^2, x^3) \tag{7}$$

to embed Minkowski space in the intersection of the $O(2,4)$ null cone N of the origin of $\mathbb{R}^6$, described by

$$N = \{(T, V, W, X, Y, Z) : T^2 + V^2 - W^2 - X^2 - Y^2 - Z^2 = 0\}, \tag{8}$$

with the hyperplane

$$H = \{(T, V, W, X, Y, Z) : V - W = 1\}. \tag{9}$$

Exercise 2 *Show that the transformation (7) is indeed an embedding of* $\mathbb{M}$ *in* $N \cap H$.

Exercise 3 *Show that on any generator of* N *with* $V - W \neq 0$ *we can find a point in* H *and hence a point of* $\mathbb{M}$.

Exercise 4 *Show that the transformation (7) is an isometry.*

The space of generators of the null cone (8) is a quadric Q in $\mathbb{RP}^5$. It is compact, and is the space which we choose to be compactified Minkowski space $\mathbb{M}^\#$. We have added points which correspond to those generators of the null cone which do not intersect H.

To discover what these extra points look like locally we must go back to the mapping (7). The non-compact space $\mathbb{M}$ is identified with a subset of the space of generators, isometrically. The extra points in $\mathbb{M}^\#$ lie on the hyperplane $V = W$, and under $O(2,4)$ this can be mapped to $V + W = 0$, on which $\Delta = 0$. So we see that the extra points are a null cone, and we have compactified Minkowski space by adding a null cone at infinity.

Finally, we complexify the space $\mathbb{M}^\#$ to obtain compactified complexified Minkowski space, $\mathbb{CM}^\#$.

3 Twistor space

Twistor space, $\mathbb{T}$, is a four dimensional complex vector space with complex coordinates

$$Z^\alpha = (Z^0, Z^1, Z^2, Z^3). \tag{10}$$

Projective twistor space, $\mathbb{PT}$, is the space of complex lines through the origin in $\mathbb{T}$, with homogeneous complex coordinates

$$Z^\alpha = (Z^0 : Z^1 : Z^2 : Z^3). \tag{11}$$

Dual twistor space, $\mathbb{T}^*$, is the space of linear functions on $\mathbb{T}$:

$$W_\alpha : \mathbb{T} \longrightarrow \mathbb{C} \tag{12}$$

$$Z^\alpha \longmapsto Z^\alpha W_\alpha \tag{13}$$

where we assume summation over the repeated index. Alternatively, if we consider a fixed dual twistor, A_α say, this defines a hyperplane through the origin in $\mathbb{T}$ given by

$$\{Z^\alpha : A_\alpha Z^\alpha = 0\}.$$

Dual twistor space is then the space of hyperplanes such as this.

4 Klein correspondence

A line in $\mathbb{PT}$ can be determined by taking the skew-symmetrised outer product of any two planes through it, as follows:

$$L_{\alpha\beta} = X_\alpha Y_\beta - X_\beta Y_\alpha,$$

and then forgetting the overall scale. So the space $\mathbb{F}_2$ is the space of these $L_{\alpha\beta}$, up to an overall scale. In coordinates, this skew-symmetric matrix is

$$L_{\alpha\beta} = \begin{pmatrix} 0 & L_{01} & L_{02} & L_{03} \\ -L_{01} & 0 & L_{12} & L_{13} \\ -L_{02} & -L_{12} & 0 & L_{23} \\ -L_{03} & -L_{13} & -L_{23} & 0 \end{pmatrix}. \tag{14}$$

The space of such matrices up to an overall scale is $\mathbb{CP}^5$. However, $L_{\alpha\beta}$ is not only skew-symmetric, it is also *simple*: it was formed by taking an outer product.

Lemma 1 *$L_{\alpha\beta}$ (skew) is also* simple *if and only if*

$$L_{\alpha\beta}L_{\gamma\delta} + L_{\alpha\gamma}L_{\delta\beta} + L_{\alpha\delta}L_{\beta\gamma} = 0. \tag{15}$$

Proof

Suppose that the skew matrix $L_{\alpha\beta}$ satisfies (15). Let P^γ and Q^δ be arbitrary twistors. Then

$$L_{\alpha\beta}L_{\gamma\delta}P^\gamma Q^\delta + L_{\alpha\gamma}L_{\delta\beta}P^\gamma Q^\delta + L_{\alpha\delta}L_{\beta\gamma}P^\gamma Q^\delta = 0. \tag{16}$$

Now let $L_{\gamma\delta}P^\gamma Q^\delta = \kappa$, $L_{\alpha\gamma}P^\gamma = X_\alpha$, and $L_{\beta\delta}Q^\delta = Y_\beta$. Then (16) becomes

$$\kappa L_{\alpha\beta} - X_\alpha Y_\beta + X_\beta Y_\alpha = 0, \tag{17}$$

and hence $L_{\alpha\beta}$ is simple. We leave the other part of the proof as an exercise.

□

Exercise 5 *Finish the proof of Lemma 1.*

Therefore the space of skew simple $L_{\alpha\beta}$ is a quadric in $\mathbb{CP}^5$. This is the *Klein correspondence.*

Theorem 1 *The space of lines in $\mathbb{PT}$ is isomorphic to $\mathbb{CM}^\#$.*

Proof

By making the following convenient choice of coordinates

$$\begin{aligned} T &= \tfrac{i}{\sqrt{2}}(L_{03} - L_{12}) \quad & V &= L_{23} + \tfrac{1}{2}L_{01} \quad & W &= L_{23} - \tfrac{1}{2}L_{01} \\ X &= \tfrac{i}{\sqrt{2}}(L_{02} - L_{13}) \quad & Y &= \tfrac{-1}{\sqrt{2}}(L_{02} + L_{13}) \quad & Z &= \tfrac{-i}{\sqrt{2}}(L_{12} + L_{03}) \end{aligned} \tag{18}$$

equation (15) becomes

$$T^2 + V^2 - W^2 - X^2 - Y^2 - Z^2 = 0$$

which is the null cone of the origin in $\mathbb{C}^6$. Recall that we defined $\mathbb{CM}^\#$ to be the space of generators of this null cone and so our quadric in $\mathbb{CP}^5$ can be thought of as compactified complexified Minkowski space. Therefore, since this quadric (15) is the space of lines in $\mathbb{PT}$, we have established the result.

□

5 Causal structure

Let x^a be the space-time point corresponding to a line $L \subset \mathbb{PT}$. For a twistor Z^α to lie on L it must satisfy two linear equations. It follows from (7) and (18) that (except when the line is given by $Z^2 = Z^3 = 0$) these two equations can be written

$$\begin{pmatrix} Z^0 \\ Z^1 \end{pmatrix} = \frac{i}{\sqrt{2}} \begin{pmatrix} x^0 + x^3 & x^1 + ix^2 \\ x^1 - ix^2 & x^0 - x^3 \end{pmatrix} \begin{pmatrix} Z^2 \\ Z^3 \end{pmatrix}. \tag{19}$$

Exercise 6 *Show this.*

Now suppose that the twistor Z^α also lies on the line corresponding to the space-time point y^a. Then

$$\begin{pmatrix} Z^0 \\ Z^1 \end{pmatrix} = \frac{i}{\sqrt{2}} \begin{pmatrix} y^0 + y^3 & y^1 + iy^2 \\ y^1 - iy^2 & y^0 - y^3 \end{pmatrix} \begin{pmatrix} Z^2 \\ Z^3 \end{pmatrix}. \tag{20}$$

We will deduce from (19) and (20) that $w^a = x^a - y^a$ is a null vector. We have

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \frac{i}{\sqrt{2}} \begin{pmatrix} w^0 + w^3 & w^1 + iw^2 \\ w^1 - iw^2 & w^0 - w^3 \end{pmatrix} \begin{pmatrix} Z^2 \\ Z^3 \end{pmatrix}, \tag{21}$$

and therefore

$$0 = \begin{vmatrix} w^0 + w^3 & w^1 + iw^2 \\ w^1 - iw^2 & w^0 - w^3 \end{vmatrix} \tag{22}$$

$$= (w^0 + w^3)(w^0 - w^3) - (w^1 + iw^2)(w^1 - iw^2), \tag{23}$$

so w^a is null as required.

Exercise 7 *Show the converse, that if $x^a - y^a$ is a null vector then the lines in $\mathbb{PT}$ corresponding to x^a and y^a intersect.*

We now have:

$\mathbb{PT}$	$\mathbb{CM}^{\#}$
complex projective line	point
intersection of lines	null-separation of points

6 Real points

The points x^a and y^a above are complex: how do we pick out real points in Minkowski space? Any twistor Z^α lying on the line in $\mathbb{PT}$ corresponding to the point x^a satisfies (19), and if x^a is real then

$$Z^0\overline{Z^2} + Z^1\overline{Z^3} + \overline{Z^0}Z^2 + \overline{Z^1}Z^3 = 0. \tag{24}$$

Exercise 8 *Show this.*

The Hermitian form

$$\Sigma(Z^\alpha) = Z^0\overline{Z^2} + Z^1\overline{Z^3} + \overline{Z^0}Z^2 + \overline{Z^1}Z^3 \tag{25}$$

divides $\mathbb{PT}$ into three regions:

$$\Sigma(Z^\alpha) > 0 \qquad \mathbb{PT}^+$$

$$\Sigma(Z^\alpha) = 0 \qquad \mathbb{PN}$$

$$\Sigma(Z^\alpha) < 0 \qquad \mathbb{PT}^-$$

Now go back to (19), and think of Z^α as a fixed point in $\mathbb{PN}$. Any two real solutions x^a and y^a of (19) must be null-separated, and so the set of solutions is a (real) null line in Minkowski space. Therefore, points in $\mathbb{PN}$ correspond to real null lines in $\mathbb{M}$. The space of these real null lines is a real 5-dimensional manifold, as can be seen from (25) or from counting them in $\mathbb{M}$. Note that not all real 5-dimensional manifolds can be extended to form a complex 3-dimensional manifold: this is a special property of the null lines in $\mathbb{M}$.

Also, a line L lies entirely in the upper half $\mathbb{PT}^+$ of projective twistor space if and only if it corresponds to a point $z^a = x^a - iy^a$ in $\mathbb{CM}^{\#}$ with y^a timelike and future-pointing. Fields defined on this part of $\mathbb{CM}^{\#}$ have *positive frequency*, an important distinction in quantum field theory which has thus been expressed geometrically in $\mathbb{PT}$.

Next, fix a line L in $\mathbb{PN}$, with its corresponding real point x^a in Minkowski space. L is a complex projective line $\mathbb{CP}^1$, the *Riemann sphere*. Twistors on L correspond to null lines through x^a. So intrinsically the line L in $\mathbb{PN}$ is the celestial sphere of the space-time point x^a.

We now have:

$\mathbb{PN}$	$\mathbb{M}^\#$
point	null line
line, intrinsically	celestial sphere of a point

7 Functions on twistor space

In this section we consider a simple example of a function on $\mathbb{PT}$ and show how to obtain from it a corresponding function on $\mathbb{CM}^\#$. Let

$$f(Z^\alpha) = \frac{1}{(A_\alpha Z^\alpha)(B_\beta Z^\beta)}. \tag{26}$$

This is well defined everywhere in $\mathbb{PT}$ except on the two planes $A_\alpha Z^\alpha = 0$ and $B_\beta Z^\beta = 0$. Strictly speaking, it is not a function, because it is homogeneous of degree -2, not 0. So it should be thought of as a local section of the *sheaf* $\mathcal{O}(2)$. We ignore this for the moment, and treat f as a function.

Choose a point $x^a \in \mathbb{CM}^\#$, and restrict f to the line in $\mathbb{PT}$ corresponding to x^a. Then from (19) $A_\alpha Z^\alpha$ becomes

$$A_0 \frac{i}{\sqrt{2}} \left[(x^0 + x^3)Z^2 + (x^1 + ix^2)Z^3)\right] + A_1 \frac{i}{\sqrt{2}} \left[(x^1 - ix^2)Z^2 + (x^0 - x^3)Z^3\right] \\ + A_2 Z^2 + A_3 Z^3,$$

which we abbreviate to $aZ^2 + bZ^3$, and similarly for $B_\beta Z^\beta$. Now

$$f(Z^\alpha|_{x^a}) = \frac{1}{(aZ^2 + bZ^3)(cZ^2 + dZ^3)}. \tag{27}$$

Note that the complex numbers a, b, c, and d depend on x^a. Next, we perform the following contour integral in the $\mathbb{CP}^1$ of the line in $\mathbb{PT}$ corresponding to x^a:

$$\phi(x^a) = \frac{1}{2\pi i} \oint \frac{Z^2 dZ^3 - Z^3 dZ^2}{(aZ^2 + bZ^3)(cZ^2 + dZ^3)} \tag{28}$$

This is done by changing the coordinates

$$\begin{pmatrix} z^0 \\ z^1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} Z^2 \\ Z^3 \end{pmatrix} \tag{29}$$

to obtain

$$\phi(x^a) = \frac{1}{2\pi i(ad - bc)} \oint \frac{z^0 dz^1 - z^1 dz^0}{z^0 z^1}. \tag{30}$$

Finally, we put $z = z^1/z^0$ to map this contour integral down onto $\mathbb{C}$, and we obtain

$$\phi(x^a) = \frac{1}{2\pi i(ad-bc)} \oint \frac{dz}{z} \tag{31}$$

$$= \frac{1}{ad-bc}. \tag{32}$$

Suppose the line K of intersection of the two planes $Z^\alpha A_\alpha = 0$ and $Z^\alpha B_\alpha = 0$ corresponds to the point k^a. Then it can be shown that

$$\frac{1}{ad-bc} = \frac{2}{(A_0B_1 - A_1B_0)(x^a - k^a)(x^b - k^b)g_{ab}}. \tag{33}$$

The significance of this result is that $\phi(x^a)$ is automatically a solution of the *wave equation*

$$\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} - \frac{\partial^2\phi}{\partial x^2} - \frac{\partial^2\phi}{\partial y^2} - \frac{\partial^2\phi}{\partial z^2} = 0$$

on the complement of the null cone of k^a.

But the relationship between the twistor function f and the field ϕ is not one-to-one. There is an equivalence class of twistor functions all of which lead to the same ϕ. We will see that this class is described by Čech cohomology, as follows.

Theorem 2

$$\check{H}^1(\mathbb{PT} \setminus K; \mathcal{O}(-2)) \cong \{\textit{analytic solutions of the wave equation on the complement of the null cone of } k^a\} \tag{34}$$

In fact, all analytic solutions of all the *zero rest mass free field* equations (which include Maxwell's equations) can be constructed in a similar way, by starting with functions $f(Z^\alpha)$ homogeneous of degree n, for various integers n.

8 A double fibration

Here we describe the geometrical framework behind the calculation of $\phi(x^a)$ from $f(Z^\alpha)$. Define

$$\mathbb{F}_1 = \{S_1 : S_1 \text{ is a one-dimensional subspace of } \mathbb{T}\}$$
$$\mathbb{F}_2 = \{S_2 : S_2 \text{ is a two-dimensional subspace of } \mathbb{T}\}$$
$$\mathbb{F}_{1,2} = \{(S_1, S_2) : S_1 \text{ and } S_2 \text{ as above with } S_1 \text{ a subspace of } S_2\}.$$

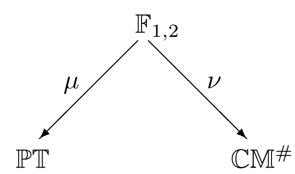
Note that $F_1 = \mathbb{PT}$ and $F_2 = \mathbb{CM}^\#$. Now define the "forgetful" maps $\mu : \mathbb{F}_{1,2} \longrightarrow \mathbb{F}_1$ given by

$$\mu(S_1, S_2) = S_1 \tag{35}$$

and $\nu : \mathbb{F}_{1,2} \longrightarrow \mathbb{F}_2$ given by

$$\nu(S_1, S_2) = S_2, \tag{36}$$

and we then have the double fibration



Exercise 9 *For Z^α a point in $\mathbb{PT}$, describe $\mu^{-1}(Z^\alpha)$, which is called the* fibre *of μ above Z^α.*

Exercise 10 *For x^a a point in $\mathbb{CM}^\#$, describe $\nu^{-1}(x^a)$, which is called the* fibre *of ν above x^a.*

Now the procedure in the previous section can be expressed as follows. Given a function $f(Z^\alpha)$ on a subset of $\mathbb{PT}$, we define its *pullback*

$$\mu^*(f)(Z^\alpha, x^a) = f(Z^\alpha), \tag{37}$$

which is a function on a subset of $\mathbb{F}_{1,2}$. Then we integrate $\mu^*(f)$ along the fibres of ν, to obtain the function $\phi(x^a)$ on a subset of $\mathbb{CM}^\#$. What are the conditions on these subsets? In general, we start by choosing

$$X \subset \mathbb{CM}^\#$$

and then select the corresponding spaces

$$Y = \nu^{-1}(X) \subset \mathbb{F}$$

and

$$T = \mu(Y) \subset \mathbb{PT}.$$

The procedure for obtaining ϕ from f will work when the fibres of $\mu|_Y$ are connected and simply connected.

9 Sheaf cohomology

This is an *extremely* brief description of sheaf cohomology. See [1] for a proper account.

Given a topological space X, a *sheaf* over X is another topological space $\mathcal{S}$ with a mapping $\pi : \mathcal{S} \to X$ satisfying

1. π is a local homeomorphism,
2. the *stalks* $\pi^{-1}(x)$ are abelian groups, and
3. the group operations are continuous.

If U is open in X we denote by $\mathcal{S}(U)$ the abelian group of *sections* of $\mathcal{S}$ over U.

As an example, take holomorphic functions on a complex manifold. Consider a point z in the manifold and a function element (f, D) such that the domain D contains z. We define the *germ* $[f, z]$ of f at z to be *all* function elements (f_i, D_i) such that $z \in D_i$ and there is a neighbourhood of z on which $f_i = f$. Then the set of germs is the sheaf $\mathcal{O}$ on the manifold.

Now we turn to Čech cohomology, still with our sheaf $\mathcal{S}$. Choose an open cover $\{U_i\}$ of X, and simplify the notation for intersections by setting

$$U_{ij} = U_i \cap U_j. \tag{38}$$

A 0-cochain is a collection

$$\{f_i \in \mathcal{S}(U_i)\}, \tag{39}$$

a 1-cochain is a collection

$$\{f_{ij} \in \mathcal{S}(U_{ij}) : f_{ij} = -f_{ji}\}, \tag{40}$$

a 2-cochain is a collection

$$\{f_{ijk} \in \mathcal{S}(U_{ij}) : f_{ijk} \text{ is skew symmetric}\}, \tag{41}$$

and similarly for a p-cochain. The set of all p-cochains forms an abelian group which we denote by $C^p(\{U_i\}; \mathcal{S})$.

We have a *coboundary map* $\delta_p : C^p(\{U_i\}; \mathcal{S}) \to C^{p+1}(\{U_i\}; \mathcal{S})$, a group homomorphism defined as follows:

$$\delta_0(\{f_i\}) = \{\rho_j f_i - \rho_i f_j\}, \tag{42}$$

$$\delta_1(\{f_{ij}\}) = \{\rho_k f_{ij} + \rho_j f_{ki} + \rho_i f_{jk}\}. \tag{43}$$

Here, ρ_i means restrict to the intersection with U_i. δ_p is defined similarly, always in such a way that the result is skew-symmetric. It is easy to check that $\delta_1 \circ \delta_0 = 0$, and in fact the skew symmetry implies that $\delta_{p+1} \circ \delta_p = 0$ for all p.

This gives us a sequence of abelian groups

$$C^0(\{U_i\}; \mathcal{S}) \xrightarrow{\delta_0} C^1(\{U_i\}; \mathcal{S}) \xrightarrow{\delta_1} C^2(\{U_i\}; \mathcal{S}) \xrightarrow{\delta_2} \dots \tag{44}$$

in which the image of δ_{p-1} is a normal subgroup of the kernel of δ_p. We define

$$\check{H}^p(\{U_i\}; \mathcal{S}) = \frac{\ker \delta_p}{\operatorname{im} \delta_{p-1}}. \tag{45}$$

Strictly speaking we should now take successive refinements of the cover $\{U_i\}$ of X and then the direct limit, in order to remove the dependence on any particular cover and obtain $\check{H}^p(X; \mathcal{S})$, the Čech cohomology of X with coefficients in the sheaf $\mathcal{S}$. But in practice it is usually possible to find a specific cover such that

$$\check{H}^p(\{U_i\}; \mathcal{S}) = \check{H}^p(X; \mathcal{S}). \tag{46}$$

Looking back at the contour integral we carried out, we can now see that the equivalence class of functions all leading to the same field ϕ is simply an element of $\check{H}^1(\{U_1, U_2\}; \mathcal{O}(-2))$, where U_1 is the complement of the plane $A_\alpha Z^\alpha = 0$ and U_2 is the complement of the plane $B_\alpha Z^\alpha = 0$.

More generally, it can be shown that

Theorem 3

$$\check{H}^1(\mathbb{PT}^+; \mathcal{O}(-2)) \cong \{\textit{positive-frequency analytic solutions of the wave equation}\} \tag{47}$$

10 Appendix

Exercise 1. Using (3), calculate

$$(ct')^2 - (x')^2 - (y')^2 - (z')^2,$$

and simplify.

Exercise 2. Firstly, note that if

$$(x^0, \frac{1}{2}(1-\Delta), -\frac{1}{2}(1+\Delta), x^1, x^2, x^3) = (y^0, \frac{1}{2}(1-\Delta), -\frac{1}{2}(1+\Delta), y^1, y^2, y^3)$$

then

$$x^a = y^a.$$

So the mapping is injective. Clearly, it is also continuous.

Exercise 3. A generator of N is the set of points

$$\{(\lambda T, \lambda V, \lambda W, \lambda X, \lambda Y, \lambda Z) : \lambda \neq 0 \in \mathbb{C}\}.$$

If $V - W \neq 0$ we can choose $\lambda = (V - W)^{-1}$, and then this point on the generator will have values of V and W satisfying $V - W = 1$.

Exercise 4. Substitute $V - W = 1$ into (6).

Exercise 5. Suppose that $L_{\alpha\beta}$ is simple. Then

$$L_{\alpha\beta} = X_\alpha Y_\beta - X_\beta Y_\alpha.$$

Now insert this expression into the left hand side of (15) and simplify.

Exercise 6. Equation (19) can be split into the two equations

$$Z^\alpha X_\alpha = 0 \qquad \text{and} \qquad Z^\alpha Y_\alpha = 0,$$

where

$$\begin{aligned} X_\alpha &= \left[-1, 0, \frac{i}{\sqrt{2}}(x^0 + x^3), \frac{i}{\sqrt{2}}(x^1 + ix^2)\right] \\ Y_\alpha &= \left[0, -1, \frac{i}{\sqrt{2}}(x^1 - ix^2), \frac{i}{\sqrt{2}}(x^0 - x^3)\right]. \end{aligned}$$

Now set

$$L_{\alpha\beta} = X_\alpha Y_\beta - X_\beta Y_\alpha,$$

and then show that

$$\frac{i}{\sqrt{2}}(L_{03} - L_{12}) = x^0, \quad \frac{i}{\sqrt{2}}(L_{02} - L_{13}) = x^1,$$

and so on, as required for (7) and (18).

Exercise 7. Suppose that $w^a = x^a - y^a$ is a null vector. Then the rows in

$$\begin{pmatrix} w^0 + w^3 & w^1 + iw^2 \\ w^1 - iw^2 & w^0 - w^3 \end{pmatrix}$$

are linearly dependent, so we can find Z^2 and Z^3 satisfying (21). Now calculate Z^0 and Z^1 from (19) or (20). Then Z^α lies on both the line corresponding to x^a and the line corresponding to y^a.

Exercise 8.

$$\begin{aligned} Z^0\overline{Z^2} &= \frac{i}{\sqrt{2}}\left[(x^0 + x^3)Z^2\overline{Z^2} + (x^1 + ix^2)Z^3\overline{Z^2}\right] \\ Z^1\overline{Z^3} &= \frac{i}{\sqrt{2}}\left[(x^1 - ix^2)Z^2\overline{Z^3} + (x^0 - x^3)Z^3\overline{Z^3}\right] \\ \overline{Z^0}Z^2 &= \frac{-i}{\sqrt{2}}\left[(x^0 + x^3)Z^2\overline{Z^2} + (x^1 - ix^2)Z^2\overline{Z^3}\right] \\ \overline{Z^1}Z^3 &= \frac{-i}{\sqrt{2}}\left[(x^1 + ix^2)Z^3\overline{Z^2} + (x^0 - x^3)Z^3\overline{Z^3}\right] \end{aligned}$$

and hence the result.

Exercise 9. $\mu^{-1}(Z^\alpha)$ is the space of all pairs $\{Z^\alpha, L\}$ where L is a line in $\mathbb{PT}$ which goes through the point Z^α. So it is the space of all pairs $\{Z^\alpha, x^a\}$, where x^a satisfies (19) for the given Z^α.

Exercise 10. $\nu^{-1}(x^a)$ is the space of all pairs $\{Z^\alpha, x^a\}$ where Z^α lies on the line in $\mathbb{PT}$ corresponding to x^a. So it is the space of all pairs $\{Z^\alpha, x^a\}$, where Z^α satisfies (19) for the given x^a.

References

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