Twistor geometry

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1 Special relativity

Implicit in our understanding of space is that when we observe a given object from different viewpoints it will appear different while in fact remaining the same. This different appearance manifests itself in mathematics when we use coordinates to describe the object. These coordinates will be those of the position vectors of various parts of the object, such as the vertices of a polyhedron. Changing the viewpoint means changing the coordinate system, by rotation, translation, and reflection. So we are familiar with the idea that the coordinates of an object can change, according to some transformation, while the object remains unchanged. In particular, under Euclidean transformations, lengths

$$\sqrt{x^2 + y^2 + z^2}$$

are invariant.

In special relativity we think about space and time together: they make *space-time*, which has four dimensions. A point of space-time is an *event* (which is a point in space and an instant in time). An observer S in space-time needs four coordinates (t, x, y, z) to describe the events she sees. Suppose another observer S' uses the coordinates (t', x', y', z'). We want to concentrate on the possibility of *relative motion* between S and S', so we make the x, y, and z axes coincide with the x', y' and z' axes when $t = 0$, and we also choose $t' = 0$ then too. Subsequently the origin of S' moves at constant speed v along the x axis of S. Note that if S is not accelerating in any way, nor is S'. In this case we call these coordinate systems inertial frames of reference.

Motivated by a desire to unify Newtonian mechanics with Maxwell's theory of electromagnetism, Einstein adopted the following two physical principles. Firstly, all inertial frames of reference are equivalent for all physical experiments. Secondly, light has the same velocity in all inertial frames of reference. Einstein discovered that as a direct consequence of these two principles, the correct transformation between S and S' is given by

$$t' = \gamma(t - vx/c^2) \qquad x' = \gamma(x - vt) \qquad y' = y \qquad z' = z, \tag{1}$$

where

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}.$$

Under these transformations, neither Euclidean lengths nor time measurements are invariant.

2 Minkowski space

Minkowski space $\mathbb{M}$ is a four dimensional real vector space. It has a metric

$$(dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2 \tag{2}$$

where $x^0 = ct, x^1 = x, x^2 = y, x^3 = z$, t being time. $\mathbb{M}$ is the space-time of special relativity, and the metric in (2) is the invariant. This metric can be positive, zero, or negative, and so at each event

$$x^a = (x^0, x^1, x^2, x^3)$$

in $\mathbb{M}$ there is a *null cone*, distinguishing these three cases.

For a mathematician, special relativity is the geometry of $\mathbb{M}$ under the transformations which are isometries of the metric (2). These transformations form the *Lorentz group* $O(1, 3)$, which we can enlarge by including translations to form the *Poincaré group.*

Exercise 1 *Show that the transformation (1) is in the Lorentz group.*

However, Maxwell's equations for electromagnetism are in fact invariant under an even larger group: the group of *conformal transformations.* This group consists of the Poincaré group together with dilations and inversions.

In conventional inversive geometry, the inversions are in circles in $\mathbb{R}^2$. For example, inversion in the unit circle is given by

$$(x, y) \longmapsto \frac{(x, y)}{x^2 + y^2}.$$

Note that this mapping is not defined at $(0, 0)$.

The corresponding inversion in Minkowski space has the analogous formula

$$x^a \longmapsto \frac{x^a}{\Delta}, \tag{3}$$

where

$$\Delta = (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2.$$

However, the transformation (3) is not well defined on all of $\mathbb{M}$, because it is singular on the null cone $\Delta = 0$. To overcome this problem we need to *compactify* Minkowski space.

The plane $\mathbb{R}^2$ can be compactified using stereographic projection from the sphere. In this process exactly one point is added to the plane, "at infinity", to obtain the sphere, which is a compact space. This is extremely important in complex analysis, for example, where it is referred to as the *Riemann sphere*. Inversions in circles are well-defined on the Riemann sphere.

Consider $\mathbb{R}^6$ with coordinates (T, V, W, X, Y, Z) and the metric

$$dT^2 + dV^2 - dW^2 - dX^2 - dY^2 - dZ^2$$

We use the mapping $\mathbb{M} \longrightarrow \mathbb{R}^6$ given by

$$x^a \longmapsto \left(x^0, \frac{1}{2}(1 - \Delta), -\frac{1}{2}(1 + \Delta), x^1, x^2, x^3\right) \tag{4}$$

to embed Minkowski space in the intersection of the null cone N of the origin of $\mathbb{R}^6$, described by

$$N = \{(T, V, W, X, Y, Z) : T^2 + V^2 - W^2 - X^2 - Y^2 - Z^2 = 0\}, \tag{5}$$

with the hyperplane

$$H = \{(T, V, W, X, Y, Z) : V - W = 1\}. \tag{6}$$

Exercise 2 *Show that the transformation (4) is indeed an embedding of* $\mathbb{M}$ *in* $N \cap H$.

Exercise 3 *Show that on any generator of* N *with* $V - W \neq 0$ *we can find a point satisfying (6) and hence a point of* $\mathbb{M}$.

Exercise 4 *Show that the transformation (4) is an isometry.*

The space of generators of the null cone (5) is a quadric Q in $\mathbb{RP}^5$. It is compact, and is the space which we choose to be compactified Minkowski space $\mathbb{M}^{\#}$. We have added points which correspond to those generators of the null cone which do not intersect H.

To discover what these extra points look like locally we must go back to the mapping (4). The non-compact space $\mathbb{M}$ is identified with a subset of the space of generators, isometrically. The extra points in $\mathbb{M}^{\#}$ lie on the hyperplane $V = W$, and under $O(2,4)$ this can be mapped to $V + W = 0$, on which $\Delta = 0$. So we achieve the compactification of Minkowski space by adding a null cone at infinity.

Finally, we complexify the space $\mathbb{M}^{\#}$ to obtain compactified complexified Minkowski space, $\mathbb{CM}^{\#}$.

3 Twistor space

Twistor space, $\mathbb{T}$, is a four dimensional complex vector space with complex coordinates

$$Z^\alpha = (Z^0, Z^1, Z^2, Z^3). \tag{7}$$

Projective twistor space, $\mathbb{PT}$, is the space of complex lines through the origin in $\mathbb{T}$, with homogeneous complex coordinates

$$Z^\alpha = [Z^0, Z^1, Z^2, Z^3]. \tag{8}$$

As usual, dual twistor space, $\mathbb{T}^*$, is the space of linear functions on $\mathbb{T}$:

$$W_\alpha : \mathbb{T} \longrightarrow \mathbb{C} \tag{9}$$

$$Z^\alpha \longmapsto Z^\alpha W_\alpha \tag{10}$$

where we assume summation over the repeated index. Alternatively, if we consider a fixed dual twistor, A_α say, this defines a hyperplane through the origin in $\mathbb{T}$ given by

$$\{Z^\alpha : A_\alpha Z^\alpha = 0\}.$$

Dual twistor space is then the space of hyperplanes such as this.

4 Klein correspondence

A line in $\mathbb{PT}$ can be determined by taking the skew-symmetrised outer product of any two planes through it, as follows:

$$L_{\alpha\beta} = X_\alpha Y_\beta - X_\beta Y_\alpha,$$

and then forgetting the overall scale. So the space $\mathbb{F}_2$ is the space of these $L_{\alpha\beta}$, up to an overall scale. In coordinates, this skew-symmetric matrix is

$$L_{\alpha\beta} = \begin{pmatrix} 0 & L_{01} & L_{02} & L_{03} \\ -L_{01} & 0 & L_{12} & L_{13} \\ -L_{02} & -L_{12} & 0 & L_{23} \\ -L_{03} & -L_{13} & -L_{23} & 0 \end{pmatrix}. \tag{11}$$

The space of such matrices up to an overall scale is $\mathbb{CP}^5$. However, $L_{\alpha\beta}$ is not only skew-symmetric, it is also *simple*: it was formed by taking an outer product.

Lemma 1 *$L_{\alpha\beta}$ (skew) is also* simple *if and only if*

$$L_{\alpha\beta}L_{\gamma\delta} + L_{\alpha\gamma}L_{\delta\beta} + L_{\alpha\delta}L_{\beta\gamma} = 0. \tag{12}$$

Proof

Suppose that the skew matrix $L_{\alpha\beta}$ satisfies (12). Let P^γ and Q^δ be arbitrary twistors. Then

$$L_{\alpha\beta}L_{\gamma\delta}P^\gamma Q^\delta + L_{\alpha\gamma}L_{\delta\beta}P^\gamma Q^\delta + L_{\alpha\delta}L_{\beta\gamma}P^\gamma Q^\delta = 0. \tag{13}$$

Now let $L_{\gamma\delta}P^\gamma Q^\delta = \kappa$, $L_{\alpha\gamma}P^\gamma = X_\alpha$, and $L_{\beta\delta}Q^\delta = Y_\beta$. Then (13) becomes

$$\kappa L_{\alpha\beta} - X_\alpha Y_\beta + X_\beta Y_\alpha = 0, \tag{14}$$

and hence $L_{\alpha\beta}$ is simple. We leave the other part of the proof as an exercise.

□

Exercise 5 *Finish the proof of Lemma 1.*

Therefore the space of skew simple $L_{\alpha\beta}$ is a quadric in $\mathbb{CP}^5$. This is the *Klein correspondence.*

Theorem 1 *The space of lines in $\mathbb{PT}$ is isomorphic to $\mathbb{CM}^\#$.*

Proof

By making the following convenient choice of coordinates

$$\begin{aligned} &T = \tfrac{i}{\sqrt{2}}(L_{03} - L_{12}) \qquad V = L_{23} + \tfrac{1}{2}L_{01} \qquad W = L_{23} - \tfrac{1}{2}L_{01} \\ &X = \tfrac{i}{\sqrt{2}}(L_{02} - L_{13}) \quad Y = \tfrac{-1}{\sqrt{2}}(L_{02} + L_{13}) \quad Z = \tfrac{-i}{\sqrt{2}}(L_{12} + L_{03}) \end{aligned} \tag{15}$$

equation (12) becomes

$$T^2 + V^2 - W^2 - X^2 - Y^2 - Z^2 = 0$$

which is the null cone of the origin in $\mathbb{C}^6$. Recall that we defined $\mathbb{CM}^\#$ to be the space of generators of this null cone and so our quadric in $\mathbb{CP}^5$ can be thought of as compactified complexified Minkowski space. Therefore, since this quadric (12) is the space of lines in $\mathbb{PT}$, we have established the result.

□

5 Causal structure

For a twistor Z^α to lie on a line L it must satisfy two linear equations. It can be shown from (4) and (15) that (except when the line is given by $Z^2 = Z^3 = 0$) these two equations can be written

$$\begin{pmatrix} Z^0 \\ Z^1 \end{pmatrix} = \frac{i}{\sqrt{2}} \begin{pmatrix} x^0 + x^3 & x^1 + ix^2 \\ x^1 - ix^2 & x^0 - x^3 \end{pmatrix} \begin{pmatrix} Z^2 \\ Z^3 \end{pmatrix} \tag{16}$$

where x^a is the space-time point corresponding to L.

Exercise 6 *Check this, by writing*

$$\begin{aligned} X_\alpha &= \left[-1, 0, \frac{i}{\sqrt{2}}(x^0 + x^3), \frac{i}{\sqrt{2}}(x^1 + ix^2)\right] \\ Y_\beta &= \left[0, -1, \frac{i}{\sqrt{2}}(x^1 - ix^2), \frac{i}{\sqrt{2}}(x^0 - x^3)\right] \end{aligned}$$

and then showing that

$$\frac{i}{\sqrt{2}}(L_{03} - L_{12}) = x^0, \quad \frac{i}{\sqrt{2}}(L_{02} - L_{13}) = x^1,$$

and so on.

Now suppose that the twistor Z^α also lies on the line corresponding to the space-time point y^a. Then

$$\begin{pmatrix} Z^0 \\ Z^1 \end{pmatrix} = \frac{i}{\sqrt{2}} \begin{pmatrix} y^0 + y^3 & y^1 + iy^2 \\ y^1 - iy^2 & y^0 - y^3 \end{pmatrix} \begin{pmatrix} Z^2 \\ Z^3 \end{pmatrix}. \tag{17}$$

We will deduce from (16) and (17) that $w^a = x^a - y^a$ is a null vector. We have

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \frac{i}{\sqrt{2}} \begin{pmatrix} w^0 + w^3 & w^1 + iw^2 \\ w^1 - iw^2 & w^0 - w^3 \end{pmatrix} \begin{pmatrix} Z^2 \\ Z^3 \end{pmatrix}, \tag{18}$$

and therefore

$$0 = \begin{vmatrix} w^0 + w^3 & w^1 + iw^2 \\ w^1 - iw^2 & w^0 - w^3 \end{vmatrix} \tag{19}$$

$$= (w^0 + w^3)(w^0 - w^3) - (w^1 + iw^2)(w^1 - iw^2), \tag{20}$$

so w^a is null as required.

Exercise 7 *Show the converse, that if $x^a - y^a$ is a null vector then the lines in $\mathbb{PT}$ corresponding to x^a and y^a intersect.*

We now have:

$\mathbb{PT}$	$\mathbb{CM}^\#$
complex projective line	point
intersection of lines	null-separation of points

6 Real points

The points x^a and y^a above are complex: how do we pick out real points in Minkowski space? Any twistor Z^α lying on the line in $\mathbb{PT}$ corresponding to the point x^a satisfies (16), and if x^a is real then

$$Z^0\overline{Z^2} + Z^1\overline{Z^3} + \overline{Z^0}Z^2 + \overline{Z^1}Z^3 = 0. \tag{21}$$

Exercise 8 *Show this.*

The Hermitian form

$$\Sigma(Z^\alpha) = Z^0\overline{Z^2} + Z^1\overline{Z^3} + \overline{Z^0}Z^2 + \overline{Z^1}Z^3 \tag{22}$$

divides $\mathbb{PT}$ into three regions:

$$\begin{array}{ll} \Sigma(Z^\alpha) > 0 & \mathbb{PT}^+ \\ \Sigma(Z^\alpha) = 0 & \mathbb{PN} \\ \Sigma(Z^\alpha) < 0 & \mathbb{PT}^- \end{array}$$

Now go back to (16), and think of Z^α as a fixed point in $\mathbb{PN}$. Any two real solutions x^a and y^a of (16) must be null-separated, and so the set of solutions is a (real) null line in Minkowski space. Therefore, points in $\mathbb{PN}$ correspond to real null lines in Minkowski space.

Next, fix a line L in $\mathbb{PN}$, with its corresponding real point x^a in Minkowski space. Twistors on L correspond to null lines through x^a. So intrinsically the line L in $\mathbb{PN}$, which is a complex projective line, is essentially the celestial sphere of the space-time point x^a.

> A complex projective line $\mathbb{CP}^1$ is intrinsically a sphere. We see this as follows. $\mathbb{CP}^1$ is the space of ratios $(z^0 : z^1)$, where z^0 and z^1 are in $\mathbb{C}$ but not both zero. Consider the two possibilities for z^0. If it is zero then $(z^0 : z^1) = (0 : 1)$, which is one point of $\mathbb{CP}^1$. If it is not zero then $(z^0 : z^1) = (1 : z^1/z^0)$, where z^1/z^0 can be any element of $\mathbb{C}$. So $\mathbb{CP}^1$ is the same as $\mathbb{C}$, with exactly one extra point. But this is simply the one-point compactification of $\mathbb{C}$.
>
> Let $\sigma : S^2\backslash\{N\} \to \mathbb{C}$ be stereographic projection from the north pole of the sphere. Then $\sigma^{-1}(z) = (1 : z) \in \mathbb{CP}^1$, and $N = (0 : 1) \in \mathbb{CP}^1$.

We now have:

$\mathbb{PN}$	$\mathbb{M}^\#$
point	null line
line, intrinsically	celestial sphere of a point

7 Functions on twistor space

In this section we consider a simple example of a function on $\mathbb{PT}$ and show how to obtain from it a corresponding function on $\mathbb{CM}^{\#}$. Let

$$f(Z^\alpha) = \frac{1}{(A_\alpha Z^\alpha)(B_\beta Z^\beta)}. \tag{23}$$

This is well defined everywhere in $\mathbb{PT}$ except on the two planes $A_\alpha Z^\alpha = 0$ and $B_\beta Z^\beta = 0$. Choose a point $x^a \in \mathbb{CM}^{\#}$, and restrict f to the line in $\mathbb{PT}$ corresponding to x^a. Then from (16) $A_\alpha Z^\alpha$ becomes

$$A_0 \frac{i}{\sqrt{2}} \left[(x^0 + x^3)Z^2 + (x^1 + ix^2)Z^3)\right] + A_1 \frac{i}{\sqrt{2}} \left[(x^1 - ix^2)Z^2 + (x^0 - x^3)Z^3\right] + A_2 Z^2 + A_3 Z^3,$$

which we abbreviate to $aZ^2 + bZ^3$, and similarly for $B_\beta Z^\beta$. Now

$$f(Z^\alpha|_{x^a}) = \frac{1}{(aZ^2 + bZ^3)(cZ^2 + dZ^3)}. \tag{24}$$

Note that the complex numbers $a, b, c,$ and d depend on x^a. Next, we perform the following contour integral in the $\mathbb{CP}^1$ of the line in $\mathbb{PT}$ corresponding to x^a:

$$\phi(x^a) = \frac{1}{2\pi i} \oint \frac{Z^2 dZ^3 - Z^3 dZ^2}{(aZ^2 + bZ^3)(cZ^2 + dZ^3)} \tag{25}$$

This is done by changing the coordinates

$$\begin{pmatrix} z^0 \\ z^1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} Z^2 \\ Z^3 \end{pmatrix} \tag{26}$$

to obtain

$$\phi(x^a) = \frac{1}{2\pi i(ad - bc)} \oint \frac{z^0 dz^1 - z^1 dz^0}{z^0 z^1}. \tag{27}$$

Finally, we put $z = z^1/z^0$ to map this contour integral down onto $\mathbb{C}$, and we obtain

$$\phi(x^a) = \frac{1}{2\pi i(ad - bc)} \oint \frac{dz}{z} \tag{28}$$

$$= \frac{1}{ad - bc}. \tag{29}$$

The significance of this result is that $\phi(x^a)$ is automatically a solution of the *wave equation* in $\mathbb{M}$. In fact, all analytic solutions of all the "zero rest mass" equations (which include Maxwell's equations) can be constructed in this way.

8 A double fibration

Here we describe the geometrical framework behind the calculation of $\phi(x^a)$ from $f(Z^\alpha)$. Define

$$\begin{aligned}
\mathbb{F}_1 &= \{S_1 : S_1 \text{ is a one-dimensional subspace of } \mathbb{T}\} \\
\mathbb{F}_2 &= \{S_2 : S_2 \text{ is a two-dimensional subspace of } \mathbb{T}\} \\
\mathbb{F}_{1,2} &= \{(S_1, S_2) : S_1 \text{ and } S_2 \text{ as above with } S_1 \text{ a subspace of } S_2\}.
\end{aligned}$$

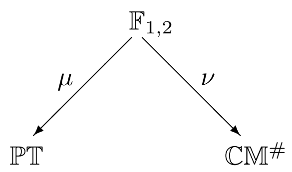
Note that $F_1 = \mathbb{PT}$ and $F_2 = \mathbb{CM}^\#$. Now define the "forgetful" maps $\mu : \mathbb{F}_{1,2} \longrightarrow \mathbb{F}_1$ given by

$$\mu(S_1, S_2) = S_1 \tag{30}$$

and $\nu : \mathbb{F}_{1,2} \longrightarrow \mathbb{F}_2$ given by

$$\nu(S_1, S_2) = S_2, \tag{31}$$

and we then have the double fibration



Exercise 9 *For Z^α a point in $\mathbb{PT}$, describe $\mu^{-1}(Z)$, which is called the* fibre *of μ above Z^α.*

Exercise 10 *For x^a a point in $\mathbb{CM}^\#$, describe $\nu^{-1}(x^a)$, which is called the* fibre *of ν above x^a.*

Now the procedure in the previous section can be expressed as follows. Given a function $f(Z^\alpha)$ on $\mathbb{PT}$, we define its "pullback" $\mu^*(f)$, which is a function on $\mathbb{F}_{1,2}$. Then we integrate $\mu^*(f)$ along the fibres of ν, to obtain the function $\phi(x^a)$ on $\mathbb{CM}^\#$.

References

[1] S A Huggett and K P Tod 1994 *An introduction to twistor theory*, second edition, Cambridge University Press.