Sequential composition of twistor diagrams

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September 17, 1998

Abstract: We regard generalised twistor diagrams as linear $SU(p,q)$–equivariant operators on tensor products of positive ladder representations. We show how a sequential composition of elementary diagrams effects the algebraic composition of the $SU(p,q)$–equivariant operators corresponding to these elementary diagrams.

1 Introduction

1.1 Background and aim of this paper

Twistor diagrams were introduced in [?] as a means to describe conformally invariant scattering amplitudes of zero–rest–mass fields of various spins in the complex analytic setting. Under the simple group $G = SU(2,2)$ — a 4–fold cover of the restricted conformal group of Minkowski space and the spinor group of $SO(2,4)$ [?, Chapter 9] — these fields form the so–called ladder representations [?] which are unitary as singular limits of the holomorphic (and anti–holomorphic) discrete series of G. They can be realised on analytic cohomology groups of holomorphic line bundles over open G–orbits of complex projective space $\mathcal{P}^3 \sim U(2,2)/(U(1) \times U(1,2))$ which in this context is called (flat) projective twistor space [?]. Various modifications of the original diagram formalism (and notation) have been proposed, in particular such that the full conformal invariance is broken, in order to write down more general scattering amplitudes [?]. We refer to [?][?] for an introduction to twistor diagrams in terms of contours and differential forms and to [?] for a cohomological account.

Here, however, we are concerned with a natural generalisation of twistor diagrams as G–equivariant operators on tensor products of positive (i.e. holomorphic) ladder representations of $G = SU(p,q)$, realised on cohomology of line bundles over $\mathcal{P}^{n-1}$ ($n = p+q$). The aim of this paper is to establish the fact that a sequential composition of twistor diagrams can be defined on the level of closed differential forms and explicit contours of integration so that the following functorial property is satisfied:

If

$$I = \otimes_{i=1}^n I_i \ , \ J = \otimes_{i=1}^n J_i \ , \ K = \otimes_{i=1}^n K_i \tag{1}$$

are tensor products of positive ladder representations of $G = SU(p,q)$ and

$$\mathcal{D}_1 \in \mathrm{Hom}_G(I,J) \ , \ \mathcal{D}_2 \in \mathrm{Hom}_G(J,K) \tag{2}$$

are G–equivariant linear operators on these tensor products, given by diagrams D_1, D_2 respectively, then the sequential composition $D_1 D_2$ effects the operator $\mathcal{D}_2 \circ \mathcal{D}_1 \in \mathrm{Hom}_G(I,K)$.

The fact that such a composition on the diagrammatic level should exist has been assumed for quite some time now, but has never been established rigorously, except in the case $p = q = 1$. Our result justifies the construction of projection operators from products (1) into G–irreducibles in diagrammatic terms by sequential composition of elementary diagrams (double twistor transforms and box diagrams). We hope, eventually, to present a complete formalism for $SU(p,q)$ in analogy with *Young tableaux* for $SU(n)$. So far, some partial results in this direction have been obtained [?]. For example it has been shown that finite linear combinations of diagrams of the form (51), so–called *box diagrams*, are sufficient to project out any given of the countably many irreducible constituents of a product of *two* positive ladder representations. Conversely, any box diagram is equivalent to a finite linear combination of such projections. For more than two factors one is led to consider a sequential concatenation (65) of elementary diagrams to form what we call a *sequential twistor diagram*. It is plausible that finite linear combinations of such sequential diagrams are sufficient to project out all irreducibles of multiple tensor products of positive ladder representations. This is known to be the case for $SU(1,1)$ where there are realisations of all the discrete series representations on analytic cohomology of degree *zero*, i.e. on spaces of holomorphic sections [**?**].

1.2 Outline of the paper

Our representations are realised on cohomology groups of holomorphic line bundles over G–orbits of projective space. The formalism of twistor diagrams, however, is primarily defined by representative meromorphic closed forms and contours of integration for these forms which avoid the variety on which they are singular. Thus two strategies are possible. Either one translates all representatives and operations on them (such as wedge product, integration etc.) into purely cohomological terms — such an approach is taken in [**?**] — or one shows the formalism to apply to preferred representatives of cohomology elements. Here we shall take this second approach, because otherwise the diagrammatic nature of the formalism is lost. We work within the Čech frame and represent cohomology elements with respect to a fixed finite Stein cover of our open orbits. In this way we only get the span of so–called *elementary states* which is dense in (the Hilbert space completion of) the respective cohomology groups (as it corresponds to the K–finite vectors of the representations for a suitable maximal compact subgroup $K \subset G$) [**?**]. Nevertheless, we believe that a generalisation of the infinite Stein cover of [**?**][1] from the case $p = q = 2$ to arbitrary p, q would enable us, without change, to apply our method of integration to arbitrary cohomology elements, represented in a preferred way according to [**?**]. We shall come back to this issue later (in subsection 2.3).

We now give an overview of our construction with its two aspects which we treat in parallel. First we have to define a cycle representing a homology class over which to integrate the meromorphic closed differential form ω whose singularity structure is represented by a sequential diagram such as (65). We can describe such a cycle as the product of standard contours (35) and (55) for the constituents of the sequential diagram. In the product we scale these contours in order to avoid the singularities of ω, choosing to enlarge them from left to right (66). In fact our descriptions and calculations are all done non–projectively, i.e. in products of $\mathbf{C}^n$ rather than $\mathcal{P}^{n-1}$ although contours and differential forms descend to products of projective spaces by integrating out the S^1–fibres of

$$\mathbf{C}^n \supset S^{2n-1} \longrightarrow \mathcal{P}^{n-1}. \tag{3}$$

This is convenient, because projectively the cycle for ω does not factor and a scaling condition such as (66) would be less easy to state. Also, in the process of integration it is essential that we can divide our variables $x_n = (x_p\,,\, x_q) \in \mathbf{C}^{p+q}$ into the first p and last q.

Second, once we have an explicit contour we integrate out ω from left to right, except for the variables x_p^{i1} $(i = 1, \ldots, k)$ on the very left of (65). Because the factors of our contour increase from left to right we can always assume that an expansion (34) or (54) exists to the right of our stage of integration. Modulo an equivalence ($\sim$) up to terms which give zero after the final integration of the variables x_p^{i1}, every integration is shown in (36) to produce the identity for a double twistor transform (31) or, in (56), a G–equivariant operator $\mathcal{D}$ (63) corresponding to a box diagram (51) (multiplied by some irrelevant non–zero constants). Thus, iteration of these two integration procedures gives the theorem (73).

The notation is mainly set up in subsection 2.2.

[1] We thank one of our referees for this reference

2 Ladder representations of $SU(p,q)$

2.1 Their cohomological realisation

The discrete series representations of semisimple Lie groups all have $\bar{\partial}$–cohomological realisations on holomorphic bundles over homogeneous spaces constructed from these groups [**?**][**?**]. Here we consider the so–called *ladder representations* of $SU(p,q)$ which are singular limits of the (anti–) holomorphic discrete series and whose cohomological realisations are particularly simple. There is a formalism using L^2 Dolbeault cohomology for indefinite metrics [**?**]. However, we shall use a Čech realisation with *elementary states* generalising the twistor framework for $SU(2,2)$ as in [**?**][**?**]. Thus we consider the natural action of $SU(p,q)$ on $\mathcal{P}_+$, one of the two open orbits of complex projective space

$$\mathcal{P}^{n-1} \sim U(p,q)/(U(1)\times U(p-1,q)) \;\; (n=p+q) \tag{4}$$

defined by

$$\mathcal{P}_+ = \{[x]\in\mathcal{P}^{n-1}\,|\,(x,x)>0\} \tag{5}$$

where $(\ ,\ )$ is the Hermitian form of signature (p,q) defining $G=SU(p,q)$. In all of the following the example of $SU(1,1)$ should be born in mind. To exemplify the construction of elementary states we start with a special case. Any choice of p linearly independent vectors a_i with $[a_i]\in\mathcal{P}_+$ $(i=1,\ldots,p)$ gives rise to a Čech representative $\prod_{i=1}^p (a_i,x)^{-1}$ of an element in

$$H^{p-1}(\mathcal{P}_+\,,\,\otimes^{-p}\mathcal{H}\,) \tag{6}$$

(where $\mathcal{H}$ is the sheaf of holomorphic sections of the hyperplane bundle restricted to $\mathcal{P}_+$ and $\mathcal{H}^{-1}$ is its dual) as follows. The sets

$$\mathcal{U}_i := \{[x]\in\mathcal{P}^{n-1}\,|\,(a_i\,,\,x)\neq 0\,\} \tag{7}$$

are Stein and thus one has a (composition of) Mayer–Vietoris connecting homomorphism(s) δ

$$\delta\colon H^0(\cap_{i=1}^p\mathcal{U}_i\,,\,\otimes^{-p}\mathcal{H}\,)\longrightarrow H^{p-1}(\cup_{i=1}^p\mathcal{U}_i\,,\,\otimes^{-p}\mathcal{H}\,) \tag{8}$$

which maps to zero sections f which can be written as

$$f = f_I - f_J \text{ (restricted to } \cap_{i=1}^p\mathcal{U}_i) \tag{9}$$

with f_I holomorphic in $\mathcal{U}_I = \cap_{i\in I}\mathcal{U}_i$ and f_J holomorphic in U_J with $I,\,J\neq\emptyset,\{1,\ldots,p\}$. Second one has an injection [**?**]

$$i\colon H^{p-1}(\cup_{i=1}^p\mathcal{U}_i\;,\;\otimes^{-p}\mathcal{H})\hookrightarrow H^{p-1}(\mathcal{P}_+\;,\;\otimes^{-p}\mathcal{H}) \tag{10}$$

by restriction of representatives. The cohomology on the right hand side is actually bigger than the image of the left hand side due to elements which blow up on $\cup_i\mathcal{U}_i - \mathcal{P}_+$. But there is a Hilbert space closure of this image within the right hand side with respect to a Hermitian inner product, discussed below. This closure is independent of our choice of $[a_i]$ $(\in\mathcal{P}_+)$ and we denote it by

$$H^{p-1}(\bar{\mathcal{P}}_+\,,\,\otimes^{-p}\mathcal{H})_{cl} \tag{11}$$

as it is the closure of the direct limit $\mathcal{U}\to\bar{\mathcal{P}}_+$ of the cohomologies over open $\mathcal{U}\supset\bar{\mathcal{P}}_+\supset\mathcal{P}_+$. It carries the unitary ladder representation $(\pi_{-p}\,,\,H_{-p})$ of $SU(p,q)$.

For any homogeneity[2] $h\in\mathbf{Z}$ an abstract Hermitian inner product $\langle\,|\,\rangle_h$ can be defined on $H_h\cong H^{p-1}(\bar{\mathcal{P}}_+\,,\,\otimes^h\mathcal{H})_{cl}$ via the so–called generalised twistor transform [**?**][**?**]

$$\mathcal{T}_h\colon H^{p-1}(\bar{\mathcal{P}}_+\,,\,\otimes^h\mathcal{H})_{cl}\cong\!\!\longrightarrow H^{q-1}(\bar{\mathcal{P}}^*_-\,,\,\otimes^{-h-n}\mathcal{H}^*)_{cl} \tag{12}$$

(which is a linear isomorphism) and a natural bilinear pairing

$$B_h\colon H^{p-1}(\bar{\mathcal{P}}^*_+\,,\,\otimes^h\mathcal{H}^*)_{cl}\otimes H^{q-1}(\bar{\mathcal{P}}^*_-\,,\,\otimes^{-h-n}\mathcal{H}^*)_{cl}\longrightarrow\mathbf{C}, \tag{13}$$

[2]The case p or $q=1$ with $h\geq 0$ requires a slight modification. One has to take quotients by global sections. But we suppress this here.

such that

$$\langle f|e\rangle_h = B_h(f^* \otimes \mathcal{T}_h(e)) \tag{14}$$

where $f \mapsto f^*$ is given by complex conjugation which takes a positive/holomorphic ladder representation to its negative/anti–holomorphic dual.

In the next subsection we generalise our example of the very special element $i \circ \delta(\prod_{i=1}^{p}(a_i, x)^{-1})$ in H_{-p} to so–called elementary states of arbitrary homogeneity h ($\in \mathbf{Z}$). On these it is easy to describe the action π_h of G and to give a generalised twistor diagram with a contour of integration for a concrete realisation of the Hermitian G–invariant scalar product $\langle\ |\ \rangle_h$. The closure of their span with respect to $\langle\ |\ \rangle_h$ gives the Hilbert spaces H_h. As a representation space, however, we can identify H_h with either of the spaces (12).

2.2 Elementary states and their Hermitian inner product

The above construction works for arbitrary (integral) powers of $\mathcal{H}$. Let a_i be as before and b_j ($j = 1, \ldots, q$) be linearly independent with $[b_j] \in \mathcal{P}_-$ ($(b_j\,,\,b_j) < 0$). We call

$$\prod_{i=1}^{p}(a_i\,,\,x)^{-\alpha_i}\prod_{j=1}^{q}(b_j\,,\,x)^{\beta_j} \ \ (\alpha_i\,,\,\beta_j \in \mathbf{N}) \tag{15}$$

or rather its image under $i \circ \delta$ in

$$H^{p-1}(\bar{\mathcal{P}}_+\,,\,\otimes^h\mathcal{H})_{cl} \ \ (h := h_1 - h_2\,;\, h_1 := \Sigma\beta_j\,,\, h_2 := \Sigma\alpha_i) \tag{16}$$

an *elementary state of homogeneity* h *based on* a_i, b_j. In order for this image not to be cohomologous to zero one needs $\alpha_i > 0$ ($i = 1, \ldots, p$), see (9). If a_i, b_j are understood to be held fixed then we can define coordinates

$$x_{\mathbf{i}} := (a_i\,,\,x) \ \ ,\ \ x_{\mathbf{p+j}} := (b_j\,,\,x) \ \ (i = 1, \ldots, p\,;\ j = 1, \ldots, q) \tag{17}$$

and we write

$$\frac{x_q^\beta}{x_p^\alpha} \ \text{ or } \ \frac{[x_q]_{h_1}}{[x_p]_{h_2}} \ \text{ with } \alpha = (\alpha_1\ \ldots, \alpha_p)\ ,\ \beta = (\beta_1\ \ldots, \beta_q) \tag{18}$$

for (representatives of) elementary states based on a_i, b_j, depending on whether we want to be specific about the multi–exponents α, β or not. (Thus we use bold face indices $x_{\mathbf{i}}$ for concrete indices whereas x_p, x_q stand for any of the variables $x_{\mathbf{1}}, \ldots, x_{\mathbf{p}}$ or $x_{\mathbf{p+1}}, \ldots, x_{\mathbf{p+q}}$, respectively, and x_n or x for either.) The action of G on these states is then induced from the natural action of G on a_i, b_j, see subsection 2.3. From now on we shall drop the distinction between elementary states and their representatives which are unique for fixed a_i, b_j if $\alpha_i > 0$.

In the following we shall fix a_i, b_j such that

$$+(a_i\,,\,a_i) = -(b_j\,,\,b_j) = 1 \tag{19}$$

and all other inner products are zero. Elementary states based on such a_i, b_j span the K–finite vectors for a corresponding choice of maximal compact $K \subset G$. For $z \in \mathcal{P}^*$ we set

$$z_{\mathbf{i}} := (\bar{z}\,,\,a_i) \ \ ,\ \ z_{\mathbf{p+j}} := -(\bar{z}\,,\,b_j) \tag{20}$$

such that

$$(x \cdot z) := (\bar{z}\,,\,x) = \sum_{k=1}^{n} x_{\mathbf{k}} z_{\mathbf{k}}\ . \tag{21}$$

We abbreviate

$$(x \cdot z)_p := \sum_{i=1}^{p} x_{\mathbf{i}} z_{\mathbf{i}} \text{ and } (x \cdot z)_q := \sum_{j=1}^{q} x_{\mathbf{p+j}} z_{\mathbf{p+j}} \tag{22}$$

and similarly

$$|x_p| := (\sum_{i=1}^{p} x_{\mathbf{i}}\bar{x}_{\mathbf{i}})^{1/2} \text{ and } |x_q| := (\sum_{j=1}^{q} x_{\mathbf{p+j}}\bar{x}_{\mathbf{p+j}})^{1/2}\ . \tag{23}$$

If we have an elementary state $e \in H_h$ and a dual elementary state $f^* \in H_h^*$ realised as

$$e = \frac{[x_q]_{h_1}}{[x_p]_{h_2}} \in H^{p-1}(\bar{\mathcal{P}}_+ \,,\, \otimes^h \mathcal{H})_{cl} \;,\; f^* = \frac{[z_q]_{g_1}}{[z_p]_{g_2}} \in H^{p-1}(\bar{\mathcal{P}}_+^* \,,\, \otimes^h \mathcal{H}^*)_{cl} \tag{24}$$

with $h = h_1 - h_2 = g_1 - g_2$ then the Hermitian inner product $\langle f|e\rangle_h$ is represented by a diagram

$$\frac{[x_q]_{h_1}}{[x_p]_{h_2}} \bullet\!\!-\!\!\!-\!\!\!-\!\!\!-\!\!\circ \frac{[z_q]_{g_1}}{[z_p]_{g_2}} \tag{25}$$

which denotes a differential form

$$e\,\omega\, f^* = \frac{[x_q]_{h_1}}{[x_p]_{h_2}} \frac{d^n x d^n z}{(x \cdot z)^r} \frac{[z_q]_{g_1}}{[z_p]_{g_2}} \,. \tag{26}$$

Originally [**?**], a twistor diagram was regarded as defining a differential form *together with* a contour or, in some cases, a small number of contours. For this diagram there is in fact only one homology class which respects the two elementary states at either end as cohomology classes, see the remark after (16), so the question does not arise yet. But in any event, we do not in this article regard the contour as being specified by the diagram. Also, one often writes $r-1$ over the line in (25) to indicate the power of the pole [**?**]. However, we omit this since it suffices for our diagrams to reflect the singularity structure of the differential forms. We choose $r = n + h$ so that we have *total homogeneity zero* in the variables x and z. If necessary we increase q to ensure $r \geq 1$. This is straightforward since the elementary states (15) for $SU(p,q)$ inject into the elementary states for $SU(p,q')$ for any $q' \geq q$, by setting $\beta_j = 0$ for $j > q$.

Now we can describe a standard $2n$–dimensional contour $\mathcal{C} \times (S^1)^{2p}$ over which to integrate the closed differential form $e\,\omega\, f^*$:

$$\mathcal{C} \times (S^1)^{2p} = \left\{ (x,z) \in \mathbf{C}^{2n} \;\middle|\; \begin{array}{l} |x_{\mathbf{i}}| = |z_{\mathbf{i}}| = \rho \;\; (\mathbf{i} = 1, \ldots, p) \\ |x_q| = 1 \,,\; z_q = e^{i\phi} \bar{x}_q ;\; \phi \in [0, 2\pi] \end{array} \right\} \tag{27}$$

with $0 < \rho < p^{-1/2}$ to avoid the zeros of the denominator of (26). Integration of $e\,\omega\, f^*$ over this contour gives a G–invariant Hermitian scalar product $\langle f|e\rangle_h$ of elementary states (times multiples of $2\pi i$). In the remaining sections we shall always use *double* twistor transforms $\mathcal{T}_{-h-n} \circ \mathcal{T}_h \sim \mathcal{I}d$ and thus dual elementary states $f^* \in H_h^* \cong H^{q-1}(\bar{\mathcal{P}}_- \,,\, \otimes^{-h-n}\mathcal{H})_{cl}$.

2.3 Twistor diagrams as $SU(p,q)$–equivariant operators

In this subsection we explain how twistor diagrams give rise to G–equivariant operators. On an elementary state such as (15) the action of an element $g \in G$ is given by

$$(a_i \,,\, x) \;,\; (b_j \,,\, x) \mapsto (g a_i \,,\, x) \;,\; (g b_j \,,\, x) \tag{28}$$

where $a_i \mapsto g a_i$ is the standard action of G on $\mathbf{C}^{p+q}$. At least for g close to the identity we can then re–expand the resulting elementary state (based on $g a_i$, $g b_j$) into an infinite sum of elementary states (16). By (9), terms with at least one of the $\alpha_i \leq 0$ in this expansion are cohomologous to zero. Similarly, for dual elementary states, realised as in (24) we take the dual action

$$(\bar{z} \,,\, a_i) \;,\; (\bar{z} \,,\, b_j) \mapsto (\bar{z} \,,\, g^d a_i) \;,\; (\bar{z} \,,\, g^d b_j) \tag{29}$$

defined by $(gc \,,\, g^d d) = (c \,,\, d)$, $\forall c, d \in \mathbf{C}^n$. Thus, by the G–invariance of $(\;,\;)$, $g^d = g \;\forall g \in G$.

A (generalised) twistor diagram D evaluated between k elementary states and k dual elementary states gives rise to complex numbers c_D which we associate with a multilinear operator $\mathcal{D}$ in the natural way:

$$\left\langle \otimes_{j=1}^k f_j |\mathcal{D}| \otimes_{i=1}^k e_i \right\rangle := c_D(\otimes_{i=1}^k e_i \,,\, \otimes_{j=1}^k f_j^*). \tag{30}$$

The fact that $\mathcal{D}$ is G–equivariant follows from the integrand being a product of powers of G–invariant inner products $(\;,\;)$ and the fact that the g–shifted contour of integration for the closed integrand is homologous to the original contour for g close to the identity.

Finally we remark that of course one would like to evaluate twistor diagrams on arbitrary cohomology elements as opposed to special elementary states with respect to a fixed finite Stein cover only. In [**?**] preferred Čech representatives of general cohomology elements are constructed with respect to a fixed infinite cover $\mathcal{U}$ for the case $p = q = 2$. The open Stein sets U_x in $\mathcal{U}$ are indexed by the points $x \in \mathcal{P}_+$ with the property that $x \notin U_x$. For arbitrary elements in (16) (with $p = 2$) preferred Čech cochains $\{f_{uv}(x)\}$ are given with $f_{uv}(x)$ defined on $U_u \cap U_v$ varying holomorphically in u, v (and x). The *evaluation* (as a space–time field) of such a cochain is by integration essentially of $f_{uv}(x)$ around u and v after restriction to the line through u and v. We believe that this construction generalises exactly to our case at hand for general elements in (16) (p arbitrary, fixed). Thus, we expect in general to have preferred representative cochains $\{f_I(x)\}$ (indexed by ordered p-tuples of p linearly independent points $\in \mathcal{P}_+$) with respect to a Stein cover $\{U_J\}$ (whose sets are indexed by all sets J of $p-1$ linearly independent points $\in \mathcal{P}_+$ and do not contain the (projective) span $\langle J \rangle$ of these points) such that $f_I(x)$ are defined on the intersection $\cap_J U_J$, where J ranges over all subsets of $p-1$ points from I, and $f_I(x)$ varies holomorphically in the points of I (and x). Evaluation of the cochain at $\langle K \rangle$ amounts to choosing I with $\langle I \rangle = \langle K \rangle$, restricting f_I to $\langle I \rangle$ and integrating around the p hyperplanes $\langle J \rangle$ within $\langle I \rangle$. To sum up, we believe that a construction of preferred representatives for general cohomology elements exists which allows evaluation by integration over the same contours as we define for elementary states. Furthermore, the fact that such an evaluation of a twistor diagram on these representatives gives G–equivariant operators independent of all further choices (up to multiplicative factors) can be inferred from the properties of the differential form associated with the diagram under the Casimir operators of G. We do not feel, however, that these issues contribute to the understanding of the *sequential composition* of diagrams and thus we shall refrain here from elaborating these points any further.

3 Double twistor transform and box diagrams

In this section we look at two special types of diagrams from which we build up a general sequential diagram in the next section: these are the *double twistor transform* and the *box diagrams*. For both of them we give standard cycles over which to integrate the closed differential forms represented by the diagrams. The choice of homology classes is in fact very small. If we assume we have elementary states at the ends of a diagram as in (25) we integrate over circles S^1 around their poles as in (27). This ensures that the integral is a functional on the cohomology groups (16) as it gives zero on coboundaries (9). The remaining part of the contour is then necessarily homologous to (35) (or to zero) in the case of the double twistor transform whereas for the box diagram there is a second generator besides (55) [**?**]. The standard contour of an arbitrary sequential diagram will then be the product of these standard contours of its constituent basic diagrams, suitably scaled to avoid all singularities.

3.1 The double twistor transform

We look at a diagram which has singularity structure

$$x \bullet \!\!-\!\!-\!\!-\!\!-\!\!-\!\!\circ\, z \!-\!\!-\!\!-\!\!-\!\!-\!\!\bullet\, u \tag{31}$$

and which thus corresponds to a differential form

$$\frac{d^n x\, d^n z\, d^n u}{(x \cdot z)^{r_1} (u \cdot z)^{r_2}} \quad \text{with } \; r_1 + r_2 = n,\; r_i \geq 1. \tag{32}$$

We assume that we have an elementary state

$$\frac{[x_q]_{h_1}}{[x_p]_{h_2}} \tag{33}$$

to the left of the diagram and to the right a function $f(u)$ which can be expanded into an absolutely convergent sum

$$f(u) = f(u_p, u_q) = \sum_{h_3=0}^{\infty} P_{h_3}(u_p) / Q_{r+h_3}(u_q) \tag{34}$$

for u_p, u_q inside a certain domain of the form $|u_p| < \rho_3|u_q|$ ($\rho_3 > 1$). Here $P_{h_3}(u_p)$ and $Q_{r+h_3}(u_q)$ denote polynomials in the variables u_p and u_q of homogeneity h_3 and $r + h_3$ ($r \geq 1$) respectively. In fact we will always have homogeneity zero in all variables ($n = h_2 - h_1 + r_1 = r + r_2$) but we do not need to impose this for the following. Our aim is to show that integration of the variables z_n, x_q and u_p over the standard $2n-$dimensional contour $\mathcal{S}$ defined by

$$\mathcal{S} = \left\{ (x_q, z, u_p) \in \mathbf{C}^{2n} \,\middle|\, \begin{array}{l} z_q = \rho_1 e^{i\phi}\bar{x}_q \,,\; u_p = \rho_2 e^{i\psi}\bar{z}_p \,, \\ |x_q| = |z_p| = 1 \,;\; \phi, \psi \in [0, 2\pi] \end{array} \right\} \tag{35}$$

with $1 < \rho_1 < \rho_2 < \rho_3$ will leave us with something *equivalent* to the elementary state (33) times the function f (34). More precisely, we want to show:

Lemma 1

$$\frac{d^p x_p}{[x_p]_{h_2}} \left[\int_{\mathcal{S}} [x_q]_{h_1} \frac{d^q x_q d^n z_n d^p u_p}{(x \cdot z)^{r_1} (u \cdot z)^{r_2}} f(u_p, u_q) \right] d^q u_q \sim \kappa \cdot \frac{[u_q]_{h_1}}{[x_p]_{h_2}} f(x_p, u_q) d^p x_p d^q u_q \ . \tag{36}$$

By equivalent ($\sim$) we mean that after integration of the variables x_p along $(S^1)^p = \times_{\mathbf{i}=1}^{p} \{|x_{\mathbf{i}}| = \rho\}$, with ρ small enough ($< p^{-1/2}$) to justify the expansion (38), we get the same differential form. (However, notice that we do *not* perform this integration.) It will actually turn out that the constant κ only depends on r_1, r_2, p and q. Thus the double twistor transform (31) is a multiple of the identity, a fact which is of course well known [**?**] and follows more or less immediately from the G–invariance of the integral. However, we choose here to give a computational proof, because much of the argument will be of use when we consider box diagrams. The abundance of variables, here and in (57), should not cloud the fact that all we are really using in the process of integration is Cauchy's theorem in one form or another, for example

$$\int_{|x_{\mathbf{k}}|=1} (z_{\mathbf{1}}\bar{x}_{\mathbf{1}} + \ldots + z_{\mathbf{m}}\bar{x}_{\mathbf{m}})^r \, x_{\mathbf{k}}^s \frac{dx_{\mathbf{k}}}{x_{\mathbf{k}}} = \pm 2\pi i \times \text{coefficient of } \bar{x}_{\mathbf{k}}^s \text{ in } (z_{\mathbf{1}}\bar{x}_{\mathbf{1}} + \ldots + z_{\mathbf{m}}\bar{x}_{\mathbf{m}})^r. \tag{37}$$

The "radial" integrations just contribute to the overall constant κ to give G–invariant expressions. They do not interest us here.

Proof: We expand

$$(x \cdot z)^{-r_1} (u \cdot z)^{-r_2} = \sum_{n_1, n_2 = 0}^{\infty} a_{r_1}(n_1)\, a_{r_2}(n_2) \frac{(x \cdot z)_p^{n_1} (u \cdot z)_q^{n_2}}{(x \cdot z)_q^{r_1 + n_1} (u \cdot z)_p^{r_2 + n_2}} \tag{38}$$

which converges absolutely in a neighbourhood of $|u_q| \leq 1$ and of $\mathcal{S} \times (S^1)^p$ chosen as above. Thus, for x_p and u_q within the described range we may interchange the integration in (36) with the expansions (34) and (38) and consider one term at a time whose relevant part looks like:

$$\begin{aligned} &\frac{d^p x_p}{[x_p]_{h_2}} \int_{\mathcal{S}} [x_q]_{h_1} \frac{(x \cdot z)_p^{n_1} (u \cdot z)_q^{n_2} P_{h_3}(u_p) d^q x_q d^n z_n d^p u_p}{(x \cdot z)_q^{r_1 + n_1} (u \cdot z)_p^{r_2 + n_2}} \\ &= \frac{d^p x_p}{[x_p]_{h_2}} \int_{\mathcal{S}} [x_q]_{h_1} \frac{(x \cdot z)_p^{n_1} (u \cdot \bar{x})_q^{n_2} P_{h_3}(\bar{z}_p) d^q x_q d^n z_n d^p u_p}{(\rho_1 e^{i\phi})^{r_1 + n_1 - n_2} (\rho_2 e^{i\psi})^{r_2 + n_2 - h_3}} \ . \end{aligned} \tag{39}$$

Now, since

$$d^p z_p d^p u_p |_{\mathcal{S}} = (\rho_2 e^{i\psi})^p \frac{dz_{\mathbf{1}}}{z_{\mathbf{1}}} \wedge \ldots \wedge \frac{dz_{\mathbf{p}}}{z_{\mathbf{p}}} \wedge d(z_{\mathbf{1}}\bar{z}_{\mathbf{1}}) \wedge \ldots \wedge d(z_{\mathbf{p-1}}\bar{z}_{\mathbf{p-1}}) \wedge \frac{de^{i\psi}}{e^{i\psi}} \tag{40}$$

and similarly for $d^q x_q d^q z_q$, integration of ϕ, ψ gives zero unless

$$r_1 + n_1 = n_2 + q \ \text{ and } \ r_2 + n_2 = h_3 + p \tag{41}$$

in which case, with $r_1 + r_2 = p + q$ (32), we get $\pm(2\pi i)^2 \times$

$$\frac{d^p x_p}{[x_p]_{h_2}} \int [x_q]_{h_1} (x \cdot z)_p^{h_3} (u \cdot \bar{x})_q^{h_3 + p - r_2} P_{h_3}(\bar{z}_p) \Big(\frac{dz_{\mathbf{1}}}{z_{\mathbf{1}}} \wedge \ldots \wedge d(x_{\mathbf{n-1}}\bar{x}_{\mathbf{n-1}})\Big) \ . \tag{42}$$

We can see now that after integration of x_p around $(S^1)^p$ we only get something possibly non–zero for $h_3 = h_2 - p$ and in this case only the monomial proportional to

$$\frac{[x_p]_{h_2}}{x_{\mathbf{1}} \ldots x_{\mathbf{p}}} \ \text{ in } \ (x \cdot z)_p^{h_3 = h_2 - p} \tag{43}$$

can contribute, by Cauchy's theorem. Parametrising further

$$z_{\mathbf{1}} = r_1 e^{i\psi_1}, z_{\mathbf{2}} = \sqrt{1-r_1^2}\, r_2\, e^{i\psi_2}, \ldots, z_{\mathbf{p}} = \sqrt{1-r_1^2} \ldots \sqrt{1-r_{p-1}^2}\, e^{i\psi_p} \tag{44}$$

we obtain

$$\begin{aligned} \pm d^p z_p d^p u_p|_{\mathcal{S}} &= (\rho_2 e^{i\psi})^p (1-r_1^2)^{p-2} (1-r_2^2)^{p-3} \ldots (1-r_{p-2}^2) \times \\ &\times \frac{de^{i\psi_1}}{e^{i\psi_1}} \wedge \ldots \wedge \frac{de^{i\psi_p}}{e^{i\psi_p}} \wedge dr_1^2 \wedge \ldots \wedge dr_{p-1}^2 \wedge \frac{de^{i\psi}}{e^{i\psi}} \end{aligned} \tag{45}$$

and integration of the *angular* variables $\psi_1, \ldots, \psi_p$ replaces the various monomials in $P_{h_3}(\bar{z}_p)$ by some constants (coming from binomial coefficients and the radial variables) times the same monomials in the variables x_p, according to (37). But as we have seen in (43), after integration of x_p only *one* such monomial can contribute. Therefore we may replace $P_{h_3}(\bar{z}_p)$ by $P_{h_3}(x_p)$ (times the corresponding constant) or indeed replace $f(u_p, u_q)$ by $f(x_p, u_q)$ (times that constant).

Similarly and again with (37), the angular integrations of x_q in (42) force

$$h_1 = n_2 = h_3 + p - r_2 = h_2 - r_2 = h_2 + r_1 - n \tag{46}$$

(i.e. total initial homogeneity zero in the variables x_n in (36)) and in this case replace $[x_q]_{h_1}$ by some constant times $[u_q]_{h_1}$.

This proves our Lemma. However, for the sake of completeness we now also perform the radial integration to show that κ does not depend on the particular elementary state $[x_q]_{h_1}/[x_p]_{h_2}$. Consider one monomial with multi–exponent $\alpha = (\alpha_1, \ldots, \alpha_p)$ from the expansion

$$(x \cdot z)_p^{n_1} = \sum_{\sum \alpha_i = n_1} \left(\begin{array}{c} n_1 \\ \alpha \end{array} \right) x_p^\alpha z_p^\alpha . \tag{47}$$

With (44) and (45), writing r_i for $r_i^2 \in [0,1]$, we obtain the integral of radial variables ($x_p^\alpha \times$)

$$\left(\begin{array}{c} n_1 \\ \alpha \end{array} \right) \int \prod_{i=1}^{p-1} r_i^{\alpha_i} (1-r_i)^{\alpha_{i+1} + \ldots + \alpha_p + p - 1 - i} dr_1 \wedge \ldots \wedge dr_{p-1} \tag{48}$$

which can be evaluated as $n_1!/(n_1+p-1)!$ independent of α, and similarly for $(u \cdot z)_q^{n_2}$. Multiplying by $a_{r_1}(n_1) a_{r_2}(n_2)$ (38) we get, with (41) and (46),

$$(-1)^{n_1+n_2} \left(\begin{array}{c} n_1 + r_1 - 1 \\ r_1 - 1 \end{array} \right) \frac{n_1!}{(n_1+p-1)!} \cdot \left(\begin{array}{c} n_2 + r_2 - 1 \\ r_2 - 1 \end{array} \right) \frac{n_2!}{(n_2+q-1)!} = (-1)^{p-r_2}/(r_1-1)!(r_2-1)!. \tag{49}$$

As there are $n+2$ angular integrations we end up with

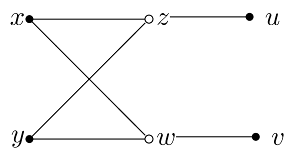
$$\kappa = \pm (2\pi i)^{2+n} (-1)^{p-r_2}/(r_1-1)!(r_2-1)! \tag{50}$$

in (36), independent of the elementary state (33).□

Thus we have verified that the double twistor transform is equivalent to (κ times) the identity in the sense of (36). To stress again the essential point: for a fixed elementary state only *one* u_p–monomial in the expansion (34) of $f(u_p, u_q)$ can possibly contribute after integration of x_p.

3.2 Box diagrams

The box diagram (with a pair of twistor transforms attached to it) has singularity structure



(51)

and corresponds to a differential form

$$\frac{d^n x\, d^n y\, d^n z\, d^n w\, d^n u\, d^n v}{(x\cdot z)^{r_1}(y\cdot z)^{r_2}(u\cdot z)^{r_3}(x\cdot w)^{s_1}(y\cdot w)^{s_2}(v\cdot w)^{s_3}} \quad \text{with} \quad \begin{array}{l} r_1+r_2+r_3 = \\ s_1+s_2+s_3 = n \end{array} \tag{52}$$

and $r_i,\ s_i \geq 1$. Here we assume that we have elementary states

$$\frac{[x_q]_{h_1}}{[x_p]_{h_2}} \ \text{and}\ \frac{[y_q]_{k_1}}{[y_p]_{k_2}} \tag{53}$$

on the left of the diagram and functions

$$\begin{array}{l} f(u) = f(u_p,u_q) = \sum_{h_3=0}^{\infty} P_{h_3}(u_p)/Q_{r+h_3}(u_q) \\ g(v) = g(v_p,v_q) = \sum_{k_3=0}^{\infty} S_{k_3}(v_p)/R_{s+k_3}(v_q) \end{array} \tag{54}$$

on the right with convergence properties as for (34). Again we only need the case where we have in fact total homogeneity zero in each variable. As the standard $4n$–dimensional contour for a box diagram we choose

$$\mathcal{B} = \left\{ \begin{array}{r|l} & |x_q| = |y_q| = |z_p| = |w_p| = 1,\ (x\cdot\bar{y})_q = 0 \\ (x_q,y_q,z,w,u_p,v_p) & z_q = \rho_1(e^{i\phi_1}\bar{x}_q + e^{i\phi_2}\bar{y}_q)\,,\ u_p = \rho_2 e^{i\phi_3}\bar{z}_p\,,\ \phi_i\in[0,2\pi] \\ \in \mathbf{C}^{4n} & w_q = \rho_1(e^{i\psi_1}\bar{x}_q + e^{i\psi_2}\bar{y}_q)\,,\ v_p = \rho_2 e^{i\psi_3}\bar{w}_p\,,\ \psi_i\in[0,2\pi] \end{array} \right\}. \tag{55}$$

We now need $2\rho_1 < \rho_2$ in order to ensure absolute convergence of the expansions of $(u\cdot z)^{r_3}$, $(v\cdot w)^{s_3}$ if we assume $|u_q| \leq 1$ as in the proof of Lemma 1, see (38). This time our aim is to show:

Lemma 2 *The integration of the product of the elementary states (53) with the form (52) and the product of f and g (54) over $\mathcal{B}$ gives a result equivalent to a* finite *linear combination of products*

$$\frac{[u_q]_{\tilde{h}_1}}{[x_p]_{\tilde{h}_2}}\frac{[v_q]_{\tilde{k}_1}}{[y_p]_{\tilde{k}_2}} \tag{56}$$

times $f(x_p,u_q)g(y_p,v_q)d^p x_p d^q u_q d^p y_p d^q v_q$.

By *equivalent* here we mean equality after integration of x_p, y_p around $(S^1)^{2p}$. It will actually turn out that $\tilde{h}_i + \tilde{k}_i = h_i + k_i$ $(i=1,2)$ but we do not need this here, see (63),(64).

Proof: We write down the essential part of one term of the integrand restricted to $\mathcal{B}$ after the expansions (38) have been made throughout:

$$\begin{aligned} &\frac{[x_q]_{h_1}}{[x_p]_{h_2}}\frac{[y_q]_{k_1}}{[y_p]_{k_2}}\frac{(x\cdot z)_p^{n_1}(y\cdot z)_p^{n_2}(x\cdot w)_p^{m_1}(y\cdot w)_p^{m_2}(u\cdot z)_q^{n_3}(v\cdot w)_q^{m_3}P_{h_3}(u_p)S_{k_3}(v_p)}{(x\cdot z)_q^{r_1+n_1}(y\cdot z)_q^{r_2+n_2}(x\cdot w)_q^{s_1+m_1}(y\cdot w)_q^{s_2+m_2}(u\cdot z)_p^{r_3+n_3}(v\cdot w)_p^{s_3+m_3}} \\ &= \frac{[x_q]_{h_1}}{[x_p]_{h_2}}\frac{[y_q]_{k_1}}{[y_p]_{k_2}}\frac{(x\cdot z)_p^{n_1}}{(\rho_1 e^{i\phi_1})^{r_1+n_1}}\frac{(y\cdot z)_p^{n_2}}{(\rho_1 e^{i\phi_2})^{r_2+n_2}}\frac{(x\cdot w)_p^{m_1}}{(\rho_1 e^{i\psi_1})^{s_1+m_1}}\frac{(y\cdot w)_p^{m_2}}{(\rho_1 e^{i\psi_2})^{s_2+m_2}}\times \\ &\times\frac{\rho_1^{n_3}(u\cdot(e^{i\phi_1}\bar{x}+e^{i\phi_2}\bar{y}))_q^{n_3}}{(\rho_2 e^{i\phi_3})^{r_3+n_3}}\frac{\rho_1^{m_3}(v\cdot(e^{i\psi_1}\bar{x}+e^{i\psi_2}\bar{y}))_q^{m_3}}{(\rho_2 e^{i\psi_3})^{s_3+m_3}}(\rho_2 e^{i\phi_3})^{h_3}P_{h_3}(\bar{z}_p)(\rho_2 e^{i\psi_3})^{k_3}S_{k_3}(\bar{w}_p). \end{aligned} \tag{57}$$

As in the case of the double twistor transform, integration of ϕ_3, ψ_3 and angular integration of the variables z_p, w_p, u_p, v_p ($\bar{z}_p$, $\bar{w}_p$) with parametrisations as in (40),(44) gives zero unless various homogeneity relations are satisfied, such as

$$\begin{gathered} r_3 + n_3 = h_3 + p \text{ and } s_3 + m_3 = k_3 + p\,, \\ n_1 + n_2 = h_3 \text{ and } m_1 + m_2 = k_3\,. \end{gathered} \tag{58}$$

Furthermore, we have equivalence to zero (~ 0) unless

$$n_1 + m_1 = h_2 - p \text{ and } n_2 + m_2 = k_2 - p\,. \tag{59}$$

From these equations we already conclude that only for finitely many n_i, m_i ($i = 1, 2, 3$) and h_3, k_3 we can possibly get something not ~ 0. In particular we can set $h_2 + k_2 = 2p + h_3 + k_3$. Integration of z_p and w_p replaces the arguments of P_{h_3} and S_{k_3} by x_p and y_p *combined* and we have to partition them again in order to obtain the desired form. Assume that we have

$$[x_p]_{h_2} = x_{\mathbf{1}}^{1+a_1} \ldots x_{\mathbf{p}}^{1+a_p} = x_{\mathbf{1}} \ldots x_{\mathbf{p}}\, x_p^a \text{ and } [y_p]_{k_2} = y_{\mathbf{1}}^{1+b_1} \ldots y_{\mathbf{p}}^{1+b_p} = y_{\mathbf{1}} \ldots y_{\mathbf{p}}\, y_p^b \tag{60}$$

with multi–exponents $a = (a_1, \ldots, a_p)$ and b. Thus, all the pairs of monomials $\bar{z}_p^\alpha$, $\bar{w}_p^\beta$ in P_{h_3} and S_{k_3} satisfying the multi–exponent relations $\alpha + \beta = a + b$ may lead to terms not ~ 0. For clarity let us consider one fixed pair of variables, $x_{\mathbf{1}}$ and $y_{\mathbf{1}}$ say. Integration of $z_{\mathbf{1}}$, $w_{\mathbf{1}}$ then replaces $\bar{z}_{\mathbf{1}}^{\alpha_1}$, $\bar{w}_{\mathbf{1}}^{\beta_1}$ by various monomials

$$\bar{z}_{\mathbf{1}}^{\alpha_1} \mapsto x_{\mathbf{1}}^\gamma y_{\mathbf{1}}^\delta \text{ and } \bar{w}_{\mathbf{1}}^{\beta_1} \mapsto x_{\mathbf{1}}^\epsilon y_{\mathbf{1}}^\zeta \text{ with } \gamma + \delta = \alpha_1 \text{ and } \epsilon + \zeta = \beta_1\,. \tag{61}$$

But the result is equivalent to zero unless $\gamma + \epsilon = a_1$ and $\delta + \zeta = b_1$. Thus we simply rewrite the resulting products

$$\frac{x_{\mathbf{1}}^\gamma y_{\mathbf{1}}^\delta}{x_{\mathbf{1}}^{1+a_1}} \frac{x_{\mathbf{1}}^\epsilon y_{\mathbf{1}}^\zeta}{y_{\mathbf{1}}^{1+b_1}} = \frac{1}{x_{\mathbf{1}} y_{\mathbf{1}}} = \frac{x_{\mathbf{1}}^{\alpha_1}}{x_{\mathbf{1}}^{1+\alpha_1}} \frac{y_{\mathbf{1}}^{\beta_1}}{y_{\mathbf{1}}^{1+\beta_1}} \tag{62}$$

and conclude that in general the exponents α_i, β_j of the denominators on the right hand side determine at most *one* monomial u_p^α and v_p^β in (54) against which the integral may be inequivalent to zero (and, *a fortiori*, $h_2 + k_2 = \tilde{h}_2 + \tilde{k}_2$). It is clear that integration of the remaining variables ϕ_i, ψ_i ($i = 1, 2$) and x_q, y_q, z_q, w_q ($\bar{x}_q$, $\bar{y}_q$) leaves us with a finite linear combination of monomials in u_q, v_q in the numerator which proves the Lemma. □

However, a thorough evaluation (which is more involved because of the restriction $(x \cdot \bar{y})_q = 0$ on $\mathcal{B}$) actually exhibits multi–exponent relations of the same kind as for the denominator. Thus one finds that integration of the box diagram over $\mathcal{B}$ is equivalent to a map of elementary states

$$\mathcal{D}:\ \frac{x_q^c}{x_p^a} \frac{y_q^d}{y_p^b} \longmapsto \sum_{\substack{\alpha + \beta = a + b \\ \gamma + \delta = c + d}} \kappa(a, b, c, d, \alpha, \beta, \gamma, \delta) \frac{x_q^\gamma}{x_p^\alpha} \frac{y_q^\delta}{y_p^\beta} \tag{63}$$

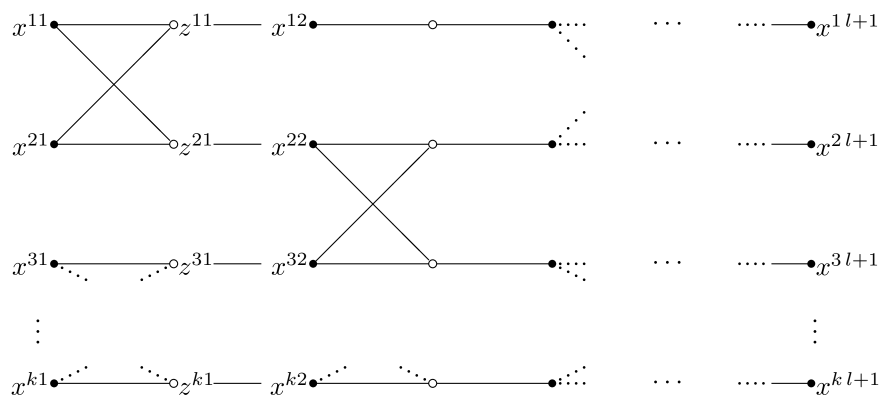
where $a, \ldots, \delta$ are multi–exponents and the constants κ are such that we get a G–invariant map. It preserves the sums of the exponents of $x_{\mathbf{i}}$ and $y_{\mathbf{i}}$ which give the weight of the corresponding K–finite vector. This is a more precise version of Lemma 2. Hence, from now on we regard box diagrams such as (51) as G–equivariant maps

$$\mathcal{D}\colon H_h \otimes H_k \longrightarrow H_{\tilde{h}} \otimes H_{\tilde{k}} \text{ with } h + k = h_1 - h_2 + k_1 - k_2 = \tilde{h} + \tilde{k} \tag{64}$$

which by Schur's lemma are multiples of the identity on each irreducible summand.

4 Sequential twistor diagrams

We now consider diagrams which are composed of double twistor transforms and box diagrams in a sequential manner, e.g.



(65)

We denote the variables on the $k(l+1)$ black vertices by x^{ij} where i and j are the indices of the row and column, respectively, where they appear. Similarly, the variables on the kl white vertices are denoted by z^{ij}. The standard contour for such a general sequential diagram will be the product of the standard contours of its constituents as described in the last section, but with radii increasing from left to right. Thus, all the double twistor transforms and box diagrams which occur in the column with index j will be integrated over contours $\mathcal{S}$ and $\mathcal{B}$ defined as in (35) and (55) with the obvious substitution of variables. In addition we replace ρ_1, ρ_2 by ρ_{2j-1} and ρ_{2j} $(j = 1, \ldots, l)$ and set

$$1 < \rho_1 <' \rho_2 < \ldots < \rho_{2l-1} <' \rho_{2l} \tag{66}$$

where $\rho_{2j-1} <' \rho_{2j}$ stands for $2\rho_{2j-1} < \rho_{2j}$. This allows us to make all the power series expansions as in the proofs of the lemmata, see (38) and the remark after (55). We may also assume, when we integrate the diagram from left to right, column by column, up to and including column $j(< l)$, that the remaining part of the diagram is a product of functions

$$f(x^{i\,j+1}) = ((x^{i\,j+1} \cdot z)_p + (x^{i\,j+1} \cdot z)_q)^{-r} \quad (r \geq 1) \tag{67}$$

which have expansions (34) with $z = z^{i\,j+1}$ or $z = z^{i\pm1\,j+1}$ depending on whether to the right of the vertex $x^{i\,j+1}$ we have a double twistor transform or a box diagram: the variable z will be restricted as in (35) or (55) to

$$|z_p| = 1 \ \ , \ \ z_q = \rho_{2j+1} e^{i\phi} \bar{x}_q^{i\,j+1} (+\rho_{2j+1} e^{i\psi} \bar{x}_q^{i\pm1\,j+1} \text{ for box}) \tag{68}$$

and

$$|x_p^{i\,j+1}| = \rho_{2j} \ (j > 0) \ , \ \ |x_q^{i\,j+1}| = 1 \ (j < l) \ . \tag{69}$$

Therefore we have absolute convergence of the expansions (34) for the functions (67) (note $(x^{i\,j+1} \cdot \bar{x}^{i\pm1\,j+1})_q = 0$ for box diagrams). On the far right $(j = l)$ we finally have (dual) elementary states

$$f_i^*(x^{i\,l+1}) = \frac{[x_p^{i\,l+1}]_{g_1^i}}{[x_q^{i\,l+1}]_{g_2^i}} \quad (i = 1, \ldots, k) \tag{70}$$

where the question of convergence does not arise. Thus we can integrate a sequential diagram column by column, from left to right, thereby replacing elementary states according to (36) or finite linear combinations of (56) depending on the part of the diagram being a double twistor transform or a box. Eventually we shall obtain a finite linear combination of products of k pairs of elementary states and dual elementary states

$$\frac{[x_q^{i\,l+1}]_{h_1^i}}{[x_p^{i1}]_{h_2^i}} \cdot \frac{[x_p^{i1}]_{g_1^i}}{[x_q^{i\,l+1}]_{g_2^i}} \ . \tag{71}$$

These we integrate around a product of circles $((S^1)^p \times (S^1)^q)^k$ defined by $|x^{i1}_{\mathbf{i}}| = \rho$ and $|x^{i\,l+1}_{\mathbf{j}}| = \rho'$ ($\mathbf{i} = 1, \ldots, p$, $\mathbf{j} = p+1, \ldots, p+q$) with $\rho < 1/\sqrt{p}$ and $\rho' < 1/\sqrt{q}$ to justify the expansions (38) also at the ends ($j = 1, l$).

We have therefore shown the following

Theorem

Let ω be a differential form defined by a $k \times l$ sequential twistor diagram (65) composed of double twistor transforms and box diagrams and let $n = p + q$ be the dimension of the vertex variables. Let $\mathcal{D}^{i\,i+1}_j$ denote the corresponding map (64) if there is a box diagram in column j and rows $(i, i+1)$ of the sequential diagram and let $\mathcal{I}^i_j$ denote the (multiple of the) identity corresponding to a double twistor transform in column j and row i. Let $\mathcal{D}_j$ be the tensor product of all these maps occurring in column j of the sequential diagram,

$$\mathcal{D}_j = \ldots \otimes \mathcal{D}^{i\,i+1}_j \otimes \ldots \otimes \mathcal{I}^{i'}_j \otimes \ldots \quad (j = 1, \ldots, l)\,, \tag{72}$$

and let $\mathcal{D} = \mathcal{D}_l \circ \ldots \circ \mathcal{D}_1$ denote the algebraic *composition of the maps $\mathcal{D}_j$.*

Then there exists a constant κ and a standard cycle $\mathcal{C}$ of dimension $2kln$ which is a product of the standard cycles for the double twistor transforms and box diagrams occurring in the sequential diagram, such that

$$\kappa \oint_{\mathcal{C} \times (S^1)^{kn}} (\Pi^k_{i=1} e_i)\, \omega\, (\Pi^k_{i=1} f^*_i) = < \otimes^k_{i=1} f_i \,|\, \mathcal{D} \,|\, \otimes^k_{i=1} e_i > \tag{73}$$

for any choice of k (dual) elementary states e_i and f^*_i $(i = 1, \ldots, k)$ on the $k + k$ exterior vertices. $\Box$

5 Concluding remarks

We have established the expected relation between sequentially composed elementary twistor diagrams and the algebraic composition of the corresponding G–equivariant maps. This required in particular a description of a suitable cycle for integration. It seems natural that such a cycle should — at least when described non–projectively — be a product of cycles for the constituent elementary diagrams. Of course, for any cycle the resulting integral is G–invariant. But it need not, even if cohomology classes of exterior states are respected, have an interpretation (30) as a G–equivariant map as it might not be defined on all of (1). One way to break full G–invariance is to allow for cycles with boundaries at prescribed subvarieties, such as the line representing space–time infinity in the classical twistor picture ($p = q = 2$) [**?**].

In representation theoretical terms our calculation was facilitated by the fact that products of positive ladder representations decompose into discrete series with finite multiplicities and thus each weight space is finite dimensional which makes the maps (64) essentially combinatorial. One might now search for standard diagrams which project out irreducibles of such products and thus provide *"Young tableaux"* for $SU(p, q)$, a study begun in [**?**]. For example the diagram (51) with $r_1 = r_2 = s_1 = s_2 = 1$ and $r_3 = s_3 = p$ projects out the lowest irreducible of $H_{-p} \otimes H_{-p}$ for any $SU(p, q)$ with $q \leq q' = 2$ and there are straightforward extensions to all q'.

There is also the possibility of composing diagrams in a vertical or, in physical terms, space–like direction. As the tensor product of positive and negative ladder representations have both continuous and discrete summands [**?**] the description of such compositions will require considerably more analysis.

Finally we would like to remark that an alternative proof of our theorem would establish the equivalence of one of the bonds

$$\cdots\!-\, z^{i\,j} \bullet\!-\!\!-\!\!-\!\!-\!\circ\, x^{i\,j+1} -\!\cdots \tag{74}$$

in the sequential diagram with summation over a complete orthonormal set of elementary states times their dual when integrated over $z^{i\,j}_p$, $x^{i\,j+1}_p$. This is straightforward in the case $p = 1$ [**?**] and for $p = 2$ such an integration can be interpreted as being over a compactified space–time of type

$S^3 \times S^1$, see (35),(55). But in any event a global description of a cycle is needed to justify power series expansions.

Acknowledgement

The authors thank Helmuth Urbantke for hospitality at the Erwin Schrödinger Institute, Vienna, where this work was carried out.