ON THE SEIFERT GRAPHS OF A LINK DIAGRAM AND ITS PARALLELS

STEPHEN HUGGETT*, IAIN MOFFATT†, AND NATALIA VIRDEE*

Abstract. Recently, Dasbach, Futer, Kalfagianni, Lin, and Stoltzfus extended the notion of a Tait graph by associating a set of ribbon graphs (or equivalently, embedded graphs) to a link diagram. Here we focus on Seifert graphs, which are the ribbon graphs of a knot or link diagram that arise from Seifert states. We provide a characterization of Seifert graphs in terms of Eulerian subgraphs. This characterization can be viewed as a refinement of the fact that Seifert graphs are bipartite. We go on to examine the family of ribbon graphs that arises by forming the parallels of a link diagram and determine how the genus of the ribbon graph of a r-fold parallel of a link diagram is related to that of the original link diagram.

1. Introduction

There is a classical way to associate a (signed) plane graph, called the Tait graph, with the diagram of a link (see Subsection 2.1). Tait graphs are a standard tool, and have found numerous applications in knot theory and graph theory. Recently, Dasbach, Futer, Kalfagianni, Lin, and Stoltzfus, in [8], extended the idea of a Tait graph by associating a set of ribbon graphs (or equivalently, embedded graphs) with a link diagram. The Tait graphs of a link diagram appear in this set of embedded graphs. One of the key advantages of Dasbach et al's idea of using non-plane graphs to describe knots is that it provides a way of encoding the crossing structure of a link diagram in the topology of the embedded graph (rather than by using signs on the edges). This idea is proving to be very useful in knot theory and it has found many recent applications. These applications include applications to the Jones and HOMFLY-PT polynomials [4, 5, 6, 8, 17, 18, 22]; Khovanov homology [3, 8]; knot Floer homology [16]; Turaev genus [1, 16, 21]; Quasi-alternating links [24]; the coloured Jones polynomial [13]; the signature of a knot [10]; the determinant of a knot [8, 9]; and hyperbolic knot theory [12].

Here we are interested in the structure of the ribbon graphs of a link diagram, and of their underlying abstract graphs. We are especially interested in the Seifert graphs of a link diagram which arise from the Seifert state of a link diagram. Seifert graphs are known to be bipartite (see [7], for example). However, as not all bipartite graphs arise from link diagrams, this does not, on its own, provide a characterization of the set of Seifert graphs of a link diagram. Here we provide a necessary and sufficient condition for a graph to be the Seifert graph of a link diagram. A well known result in graph theory states that a plane graph is bipartite if and only if its dual is Eulerian (see [2], for example). Our characterization (in Theorem 1) of Seifert graphs will be stated in terms of the dual concept of Eulerian graphs.

We then go on to determine the operation on Tait graphs which corresponds to forming the r-fold parallel of a link diagram. We conclude by applying this result to finding the genus of the

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*School of Computing and Mathematics, University of Plymouth, PL4 8AA, Devon, UK.;
s.huggett@plymouth.ac.uk, natalia.virdee@plymouth.ac.uk.
† Department of Mathematics and Statistics, University of South Alabama, Mobile, AL 36688, USA;
imoffatt@jaguar1.usouthal.edu.

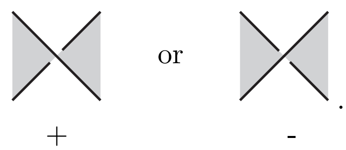
ribbon graph of a r-fold parallel of a link diagram in terms of the genus of the ribbon graph of the original link diagram.

2. The graphs of knot and link diagrams

In this section we provide an overview of the graphs and embedded graphs of a link diagram and we will describe their relation to each other. We will assume a familiarity with basic knot theory and graph theory.

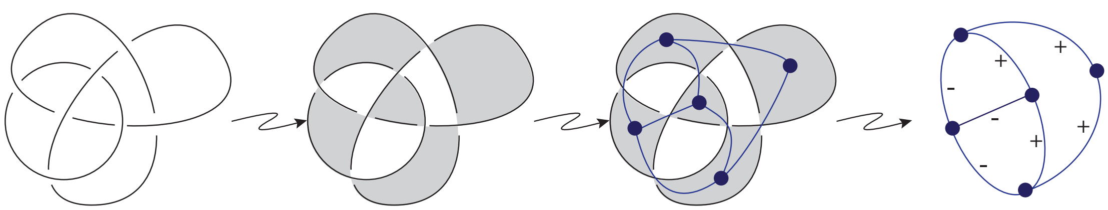
2.1. Tait graphs. Throughout this paper we will always assume that link diagrams consist of one component, that is, if $D \subset S^2$ is a link diagram, then each component of $S^2 - D$ is a disc. The components of $S^2 - D$ are called the *regions* of D. Since we prefer to work with graphs embedded in S^2 rather than $\mathbb{R}^2$, we will slightly abuse the language and refer to a graph (cellularly) embedded in S^2 as a *plane* graph.

Let $D \subset S^2$ be a link diagram. A *checkerboard colouring* of D is an assignment of the colour black or white to each region of D in such a way that adjacent regions have different colours. The *Tait sign* of a crossing in a checkerboard coloured link diagram is an element of $\{+, -\}$ which is assigned to the crossing according to the following scheme:



A *Tait graph* $\mathbb{T}(D)$ is a signed plane graph constructed from D as follows: checkerboard colour the link diagram, place a vertex in each black region and add an edge between two vertices whenever the corresponding regions of D meet at a crossing. Finally, weight each edge of the graph with the Tait sign of the corresponding crossing.

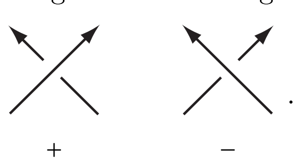
Example 1. This example illustrates the construction of a Tait graph $\mathbb{T}(D)$ of a link diagram D.



Since there are two possible checkerboard colourings of D, every diagram D has exactly two associated Tait graphs. It is well known and easily seen that the two Tait graphs associated with a link diagram are duals of each other (we refer a reader who is unfamiliar with duality forward to Subsection 2.4 for its definition). Here it should be noted that duality acts on the edge weights by switching the sign. That is, if e is an edge in a signed embedded graph G with sign m_e, then the edge e^* in G^* that corresponds to e will have sign $-m_e$.

In addition to the Tait sign, we will also need to make use of the oriented sign of a crossing.

Let D be an oriented link diagram. The *oriented sign* of a crossing of D is an element of $\{+, -\}$ which is assigned to the crossing according to the following scheme:



An *bi-weighted Tait graph*, $\mathbb{T}_\sigma(D)$, of an oriented link diagram D is a plane graph with edge weights in $\{+,-\}\times\{+,-\}$. The bi-weighted Tait graph is formed in the same way as the Tait graph except that the weight (m_e, σ_e) is assigned to each edge e, where m_e is the Tait sign of the crossing corresponding to e and σ_e is its oriented sign.

Just as with Tait graphs, each link diagram gives rise to two bi-weighted Tait graphs. Moreover, with an appropriate action of duality on the edge weights, these two bi-weighted Tait graphs are duals. If e is an edge of weight (m_e, σ_e) in an embedded graph G, then the edge e^* in G^* that corresponds to e will have sign $(-m_e, \sigma_e)$. With this action of duality on the edge weights it follows that if G and H are the two bi-weighted Tait graphs associated with an oriented link diagram then $G^* = H$.

2.2. **Ribbon graphs and their representations.** An *embedded graph* $G = (V(G), E(G)) \subset \Sigma$ is a graph drawn on a surface Σ in such a way that edges only intersect at their ends. The arcwise-connected components of $\Sigma\backslash G$ are called the *regions* of G. If each of the regions of an embedded graph G is homeomorphic to a disc we say that G is a *cellularly embedded graph*, and its regions are called *faces*. A *plane graph* is a graph that is cellularly embedded in the sphere.

Two embedded graphs, $G \subset \Sigma$ and $G' \subset \Sigma'$ are said to be *equal* if there is a homeomorphism from Σ to Σ' that sends G to G'. As is standard, we will often abuse notation and identify an embedded graph with its equivalence class under equality.

Tait graphs are edge-weighted graphs cellularly embedded in a sphere. In this paper we are interested in higher genus analogues of Tait graphs. These are edge-weighted embedded graphs that arise from link diagrams. It will be particularly convenient to use the language of ribbon graphs to describe these cellularly embedded graphs.

Definition 1. A *ribbon graph* $G = (V(G), E(G))$ is a (possibly non-orientable) surface with boundary represented as the union of two sets of topological discs, a set $V(G)$ of *vertices*, and a set of *edges* $E(G)$ such that:

(1) the vertices and edges intersect in disjoint line segments;
(2) each such line segment lies on the boundary of precisely one vertex and precisely one edge;
(3) every edge contains exactly two such line segments.

Ribbon graphs are considered up to homeomorphisms of the surface that preserve the vertex-edge structure.

A ribbon graph is said to be *orientable* if it is orientable as a surface. Here we will only consider orientable ribbon graphs. The *genus*, $g(G)$, of a ribbon graph is the genus of G as a surface. In addition we let $p(G)$ denote the number of boundary components of G, $k(G)$ the number of its connected components, $e(G) := |E(G)|$ and $v(G) := |V(G)|$.

Ribbon graphs are easily seen to be equivalent to cellularly embedded graphs. Intuitively, if G is a cellularly embedded graph, a ribbon graph representation results from taking a small neighbourhood of the cellularly embedded graph G. On the other hand, if G is a ribbon graph, we simply sew discs into each boundary component of the ribbon graph to get the desired surface.

A *spanning ribbon subgraph* of G is a ribbon graph H which can be obtained from G by deleting some edges. We will often use coloured arrows on the boundary of H to record where these deleted edges were.

Definition 2. An *arrow-marked ribbon graph* $\overrightarrow{G}$ consists of a ribbon graph G equipped with a collection of coloured arrows, called *marking arrows*, on the boundaries of its vertices. The marking arrows are such that no marking arrow meets an edge of the ribbon graph, and there are exactly two marking arrows of each colour.

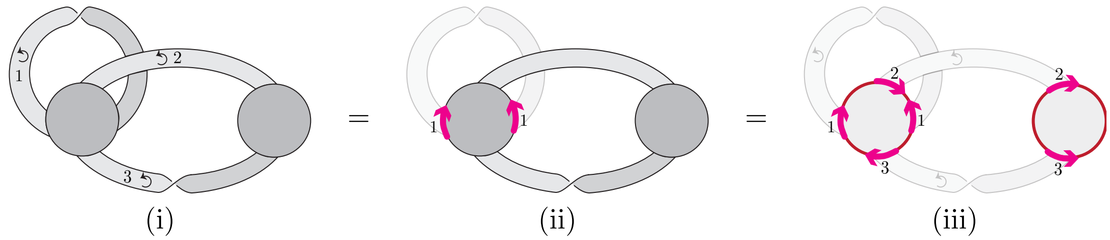
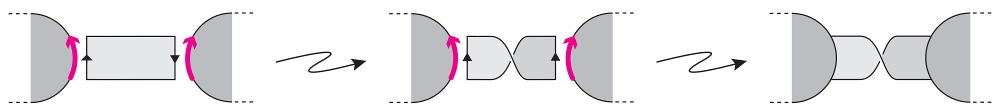


FIGURE 1. Realizations of a ribbon graph

Two arrow-marked ribbon graphs are considered to be equivalent if one can be obtained from the other by reversing the direction of all of the marking arrows which belong to some subset of colours.

A ribbon graph can be obtained from an arrow-marked ribbon graph by adding edges in a way prescribed by the marking arrows, as follows: take a disc (this disc will form the new edge) and orient its boundary arbitrarily. Add this disc to the ribbon graph by choosing two non-intersecting arcs on the boundary of the disc and two marking arrows of the same colour, and then identifying the arcs with the marking arrows according to the orientation of the arrow. The disc that has been added forms an edge of a new ribbon graph. This process is illustrated in the diagram below, and an example of an arrow-marked ribbon graph and the ribbon graph it describes is given in Figures 1(i) and (ii).



The above construction shows that an arrow-marked ribbon graph describes a ribbon graph. Conversely, every ribbon graph can be described as an arrow-marked spanning ribbon subgraph. To see why this is, suppose that G is a ribbon graph and $B \subseteq E(G)$. To describe G as an arrow-marked ribbon graph $\overrightarrow{G-B}$, start by arbitrarily orienting each edge in B. This induces an orientation on the boundary of each edge in B. For each $e \in B$, place an arrow on each of the two arcs where e meets vertices of G, the directions of these arrows following the orientation of the boundary of e. Colour the two arrows with e, and delete the edge e. This gives an arrow-marked ribbon graph $\overrightarrow{G-B}$. Moreover, the original ribbon graph G can be recovered from $\overrightarrow{G-B}$ by adding edges to $\overrightarrow{G-B}$ as prescribed by the marking arrows.

Note that if G is a ribbon graph and F is any spanning ribbon subgraph, then there is an arrow-marked ribbon graph $\overrightarrow{F}$ which describes G.

Every ribbon graph G has a representation as an arrow-marked ribbon graph $\overrightarrow{V(G)}$, where the spanning ribbon subgraph consists of the vertex set of G. In such cases, to describe G it is enough to record only the marked boundary cycles of the vertex set (to recover the vertex set, just place each cycle on the boundary of a disc). Thus a ribbon graph can be presented as a set of cycles with marking arrows on them. In such a structure, there are exactly two marking arrows of each colour. Such a structure is called an *arrow presentation*. A ribbon graph can be recovered from an arrow presentation by regarding the marked cycles as boundaries of discs, giving an arrow-marked ribbon graph. To describe this more formally:

Definition 3. An *arrow presentation* of a ribbon graph consists of a set of oriented (topological) circles (called *cycles*) that are marked with coloured arrows, called *marking arrows*, such that there are exactly two marking arrows of each colour.

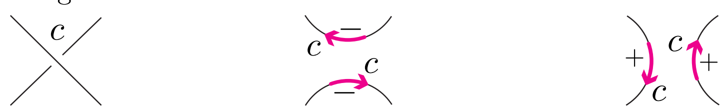
An example of a ribbon graph with its arrow presentation is given in Figure 1(i) and (iii).

Two arrow presentations are considered equivalent if one can be obtained from the other by reversing pairs of marking arrows of the same colour.

If weights are associated to the arrows, then an arrow presentation describes an edge-weighted ribbon graph.

2.3. **The ribbon graphs of a link diagram.** In [8], Dasbach et. al. extended the concept of a Tait graph by describing how a set of ribbon graphs can be associated to a link diagram. The ribbon graphs of a link diagram have proved to be very useful in knot theory. Here we are interested in certain special members of this set of ribbon graphs. In addition to the Tait graphs of a link diagram, we are particularly interested in the all-A, all-B and Seifert ribbon graphs. We now define these ribbon graphs here.

Let $D \subset S^2$ be a checkerboard coloured link diagram. Assign a unique label to each crossing of D. A *marked A-splicing* or a *marked B-splicing* of a crossing c is the replacement of the crossing with one of the following two schemes:



A crossing in D. A marked A-splicing. A marked B-splicing.

Notice that we decorate the two arcs in the splicing with signed, coloured arrows, chosen to be consistent with an arbitrary orientation of the sphere S^2. The colour of the arrows is determined by the colour of the crossing, and the signs are determined by the choice of splicing.

A *state* s of a link diagram is an assignment of a marked A- or B-splicing to each crossing. Observe that a state is precisely an arrow presentation of a ribbon graph. We will denote the ribbon graph corresponding to the state s by $\mathbb{G}(D,s)$. We say that $\mathbb{G}(D,s)$ is a *ribbon graph of the link diagram D*. We let $\mathcal{G}(D)$ denote the set of ribbon graphs of D, so that

$$\mathcal{G}(D) := \{\mathbb{G}(D,s) \mid s \text{ is a state of } D\}.$$

$\mathcal{G}(D)$ is independent of the choice of checkerboard colouring used in its construction.

If D is an *oriented* link diagram we may repeat the above construction of signed ribbon graphs, but with weights (m_c, σ_c), where m_c is the Tait sign of the crossing c and σ_c is its oriented sign. If s is such a state, we denote the ribbon graph by $\mathbb{G}_\sigma(D,s)$, and we will denote the set of ribbon graphs obtained in this way by $\mathcal{G}_\sigma(D)$.

We will now describe some special elements of $\mathcal{G}(D)$.

Tait graphs: Let $\mathbb{T}(D)$ denote a Tait graph of D. Then $\mathbb{T}(D) \in \mathcal{G}(D)$, and $\mathbb{T}(D) = \mathbb{G}(D,s)$, where the state s is obtained either by choosing an A-splicing at each $-$ crossing and a B-splicing at each $+$ crossing, or by choosing a B-splicing at each $-$ crossing and an A-splicing at each $+$ crossing.

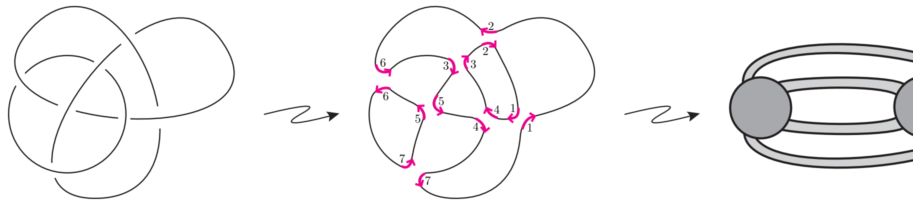
Similarly, the bi-weighted Tait graph $\mathbb{T}_\sigma(D)$ is an element of $\mathcal{G}_\sigma(D)$. Again we have $\mathbb{T}_\sigma(D) = \mathbb{G}_\sigma(D,s)$, where s is constructed as for $\mathbb{T}(D)$.

Note that in this construction, the states are chosen so that the curves will always follow the black faces, or will always follow the white faces, of the checkerboard coloured link diagram.

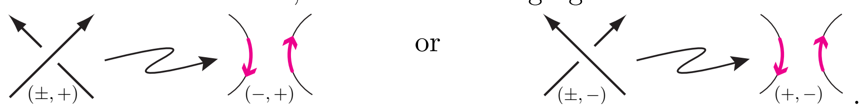
As plane graphs and genus zero ribbon graphs are equivalent, we will move freely between the realization of a Tait graph as a plane graph and as a genus zero ribbon graph. This should cause no confusion.

The all-A and all-B ribbon graphs: The *all-A ribbon graph* is defined by $\mathbb{A}(D) = \mathbb{G}(D,s)$, where s is the state obtained by choosing an A-splicing at each crossing. Similarly, the *all-B ribbon graph* is defined by $\mathbb{B}(D) = \mathbb{G}(D,s)$, where s is the state obtained by choosing a B-splicing at each crossing.

Example 2. This example illustrates the the construction of $\mathbb{A}(D)$.



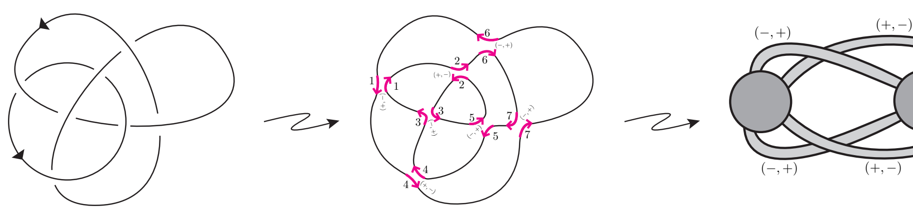
The Seifert ribbon graph: The Seifert ribbon graph $\mathbb{S}(D)$ is obtained by choosing the splicing that is consistent with the orientation, as in the following figure:



Thus if c is a crossing with oriented sign +, then choose the A-splicing at c, while if c has oriented sign − then choose the B-splicing at c.

Observe that the edge weights, $(+,-)$ and $(-,+)$, of a Seifert graph depend only on the oriented sign of the corresponding crossing in the original link diagram. This means that the construction of the Seifert graph with weights in $\{(+,-),(-,+)\}$ does not depend upon the checkerboard colouring, as one would expect.

Example 3. This example illustrates the the construction of $\mathbb{S}(D)$.



2.4. **Dual graphs.** The construction of the *dual*, G^*, of a cellularly embedded graph, $G \subset \Sigma$, is well known: form G^* by placing one vertex in each face of G and embed an edge of G^* between two vertices whenever the faces of G they lie in are adjacent. In particular, if G has k components, $G_1, \dots, G_k$, and is cellularly embedded in a surface, then each component of the graph is cellularly embedded in a connected component of the surface, and therefore duality acts disjointly on components of the graph: $(G)^* = G_1^* \cup \cdots \cup G_k^*$.

We will also need to form duals of non-cellularly embedded graphs. Since the properties of duality depend upon whether or not a graph is cellularly embedded, we will denote the dual of a not necessarily cellularly embedded graph by $G^{\circledast}$. The embedded graph $G^{\circledast}$ is formed just as the dual of a cellularly embedded graph is formed, but by placing a vertex in each region of G, rather than each face. It is important to note that, in general, $(G^{\circledast})^{\circledast} \neq G$.

There is a natural bijection between the edges of G and the edges of G^* (or of $G^{\circledast}$). We will generally use this bijection to identify the edges of G and the edges of G^*. However, at times we will be working with $G \cup G^*$ so to avoid confusion we will use e^* to denote the edge of G^* that corresponds to the edge e of G, and adopt a similar convention for sets of edges.

Observe that G^* has a naturally cellular embedding in Σ, and that there is a natural (cellular) immersion of $G \cup G^*$ where each edge of G intersects exactly one edge of G^* at exactly one point. We will call this immersion the *standard immersion* of $G \cup G^*$.

Duality has a particularly neat description in the language of ribbon graphs. Let $G = (V(G), E(G))$ be a ribbon graph, which we can regard G as a punctured surface. By filling in the punctures using a set of discs denoted $V(G^*)$, we obtain a surface without boundary Σ. The *dual* of G is the ribbon graph $G^* = (V(G^*), E(G))$.

Suppose now that $\overrightarrow{G}$ is an arrow-marked ribbon graph, so that $\overrightarrow{G}$ is a ribbon graph G with labelled arrows on its vertices. Then in the formation of G^* just described, the boundaries of the vertices of G and G^* intersect, and therefore the marking arrows on $\overrightarrow{G}$ induce marking arrows on G^*. The dual $\overrightarrow{G}^*$ of an arrow-marked ribbon graph $\overrightarrow{G}$ is the dual of its underlying ribbon graph equipped with the induced marking arrows.

2.5. **Partial duals and the ribbon graphs of link diagrams.** Partial duality, introduced by Chmutov in [6], is an extension of the concept of the dual of a cellularly embedded graph. Loosely speaking, a partial dual of a graph is obtained by forming the dual of a graph only at a given subset of edges. Partial duality has found a number of applications in graph theory, physics and knot theory (for example see [6, 11, 15, 20, 22]). Here we are interested in the motivating application of partial duality: just as geometric duality related the two Tait graphs of a link diagram, partial duality relates the ribbon graphs of a link diagram. In this section we describe partial duality and give an overview of its application to link diagrams.

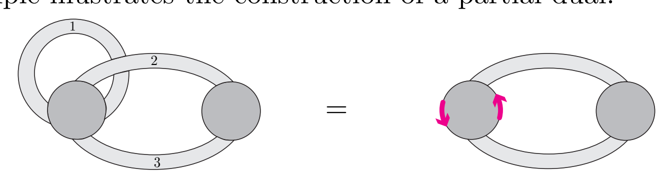
We will use the construction of a partial dual from [19]. The idea behind this construction is that given a set A of edges the partial dual with respect to A can be formed by 'hiding' the edges not in A and replacing them with marking arrows, forming the dual of the resulting arrow marked ribbon graph, and then revealing the hidden edges. We refer the reader to [6] for the original definition of a partial dual and for further details and examples of partial duals.

Definition 4. Let G be a ribbon graph, $A \subseteq E(G)$ and $A^c = E(G) - A$. Then the *partial dual*, G^A, of G with respect to A is constructed in the following way.

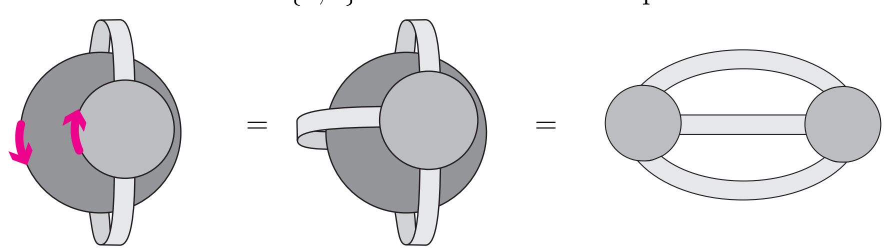
(1) Present G as the arrow-marked ribbon graph $\overrightarrow{G - A^c}$.
(2) Take the dual of $G - A^c$. The marking arrows on $\overrightarrow{G - A^c}$ induce marking arrows on $(G - A^c)^*$.
(3) G^A is the ribbon graph corresponding to the arrow-marked ribbon graph $\overrightarrow{(G - A^c)^*}$.

We will use the convention that $A^c := E(G) - A$ throughout this paper.

Example 4. This example illustrates the construction of a partial dual.



G with $A = \{2, 3\}$. Step 1.



Step 2. G^A.

There is a natural bijection between the edges of G and of G^A. We will usually use this bijection to identify the edges of G with the edges of its partial dual.

We will need the following basic properties of partial duality. The first five properties are from [6] and the sixth is from [20].

Proposition 1. *Let G be a ribbon graph and $A, B \subseteq E(G)$. Then*

(1) $G^{\varnothing} = G$;
(2) $G^{E(G)} = G^*$, *where G^* is the dual of G;*
(3) $(G^A)^B = G^{A\Delta B}$, *where $A\Delta B := (A\cup B)\backslash(A\cap B)$ is the symmetric difference of A and B;*
(4) *G is orientable if and only if G^A is orientable;*
(5) *partial duality acts disjointly on connected components;*
(6) *If G is orientable, then $g(G^A) = \frac{1}{2}\left(2k(G) + e(G) - p(G - A^c) - p(G - A)\right)$.*

Recall that the two (bi-weighted) Tait graphs of a link diagram are related by duality. Analogously, partial duality relates all of the ribbon graphs of a link diagram. In fact, the following result holds.

Proposition 2. *A ribbon graph represents a link diagram if and only if it is a partial dual of a plane graph.*

Partial duality provides a natural construction for the various ribbon graphs associated with a link diagram described in Section 2.3. Let G be an edge weighted ribbon graph and $A \subseteq E(G)$. Then G^A is also an edge weighted ribbon graph. The edge-weights of G^A are determined as follows:

- if an edge e of G has weight $m_e \in \{+,-\}$, then the corresponding edge of G^A has weight $-m_e$ if $e \in A$, and has weight m_e if $e \notin A$.
- if an edge e of G has weight $(m_e, \sigma_e) \in \{+,-\} \times \{+,-\}$, then the corresponding edge of G^A has weight $(-m_e, \sigma_e)$ if $e \in A$, and has weight (m_e, σ_e) if $e \notin A$.

With this action of partial duality on the edge-weights we have the following proposition.

Proposition 3. *Let D be an oriented link diagram. Then*

(1) *all of the ribbon graphs of the link are partial duals of either of the Tait graphs $\mathbb{T}(D)$;*
(2) $\mathbb{A}(D) = \mathbb{T}(D)^A$, *where A is the set of $+$ weighted edges of $\mathbb{T}(D)$;*
(3) $\mathbb{B}(D) = \mathbb{T}(D)^A$, *where A is the set of $-$ weighted edges of $\mathbb{T}(D)$;*
(4) $\mathbb{S}(D) = \mathbb{T}_\sigma(D)^A$, *where A is the set of all $(+,+)$ and $(-,-)$ weighted edges of $\mathbb{T}_\sigma(D)$.*

A proof of the first three statements can be found in [6] or [20], and the proof of the fourth is similar and is therefore omitted.

3. Seifert Graphs

In this section we focus on the structure of Seifert graphs. It is well known that for any link diagram D, its Seifert ribbon graph $\mathbb{S}(D)$ is bipartite (see [7], for example). Although biparticity is a necessary condition for a graph to be the Seifert graph of a link diagram, it is easily seen that it is not sufficient. Here we provide a necessary and sufficient condition for a graph to be the Seifert graph of a link diagram. A plane graph is bipartite if and only if its dual is Eulerian (see [2], for example), and our characterization of Seifert graphs will be stated in terms of the dual concept of Eulerian graphs.

Being bipartite or Eulerian is a property of abstract graphs rather than embedded graphs (in the sense that the properties are independent of how the graph is drawn in a surface). Accordingly, we will need to work with the abstract graphs of a link diagram. We will say that two embedded graphs G and H are *equivalent as abstract graphs*, written $G \cong H$, if G and H are drawings of the same (abstract) graph. An abstract graph G is a *graph of a link diagram D* if $G \cong \mathbb{G}$, for some $\mathbb{G} \in \mathcal{G}(D)$. We note that, here, the graph of a link diagram is not signed. Finally, we say that an (abstract) graph $\boldsymbol{S}(D)$ is *Seifert graph* of a link diagram D if $\boldsymbol{S}(D) \cong \mathbb{S}(D)$.

We will prove the following characterization of Seifert graphs.

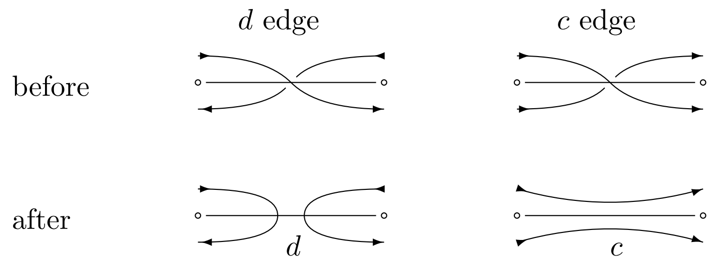


FIGURE 2. c and d edges before and after resolution of a crossing

Theorem 1. *A graph G is the Seifert graph of a link diagram D if and only if*

$$G \cong [(H \cup H^*) - (A^c \cup A^*)]^{\circledast} =: \Phi^{\circledast},$$

where H is plane, $H \cup H^*$ has the standard immersion, and each component of Φ is Eulerian. Moreover, A is the set of $(\pm,\pm)$-weighted edges in the Tait graph $\mathbb{T}_\sigma(D)$.

This theorem will follow from a careful analysis of how the Seifert graph is formed. This analysis revolves around an algorithm for obtaining the Seifert graph from a Tait graph which does not require the use of partial duals, *i.e.* we provide a way to recover $\boldsymbol{S}(D)$ from $\mathbb{T}(D)$ without having to construct the ribbon graph $\mathbb{S}(D)$.

To obtain this algorithm, we follow the steps of the Seifert algorithm on the link diagram D, while examining how each step of this algorithm affects the graph $\mathbb{T} := \mathbb{T}(D)$. Since Seifert graphs are not signed, we will ignore the edge weights of Tait graphs throughout this section. We begin by resolving the crossings of our oriented link diagram D, by following the orientation. Each crossing of D corresponds to one edge of the graph $\mathbb{T}$, and when resolving the given crossing of our link diagram we pay careful attention to what happens to the corresponding edge of $\mathbb{T}$.

At each crossing of D there are two regions corresponding to vertices of $\mathbb{T}$, which we will refer to as the black regions. We can see in Figure 2 that, depending on the orientation of the components of the link D, when we resolve a crossing of D we either separate the black regions, or we merge them together. If the regions are separated, the edge corresponding to that crossing is deleted, while if the regions are merged together the edge corresponding to that crossing is contracted. We therefore label the edges of $\mathbb{T}$ with the letters c for contraction and d for deletion.

It is these c and d labels that carry the information about the link orientation. If we change the orientation of all the components of D then the labels will not change. If, however, we only change the orientation of one of the components, then at a given crossing involving that component (other than a self-crossing) the label of the corresponding edge of $\mathbb{T}$ will change from c to d or vice versa.

If D has k components, there are 2^{k-1} possible $\{c,d\}$ colourings. Neither $\mathbb{T}$ nor $\boldsymbol{S} := \boldsymbol{S}(D)$ carry the under- and over-crossing information, which is completely independent of the $\{c,d\}$ colouring. Many different links will therefore correspond to the same $\mathbb{T}$, and even more to the same $\boldsymbol{S}$. $\mathbb{T}$ and $\boldsymbol{S}$ have the same edges, and so a $\pm$ colouring of either will do for both.

Our next task is to characterize those $\{c,d\}$ labellings of the edges of an arbitrary plane graph which come from link diagrams using the above process.

Let E_c be the set of edges of $\mathbb{T}$ coloured c, and let V_c be the set of vertices of $\mathbb{T}$ adjacent to a c edge. Let $C = (V_c, E_c)$. The graph is not necessarily connected.

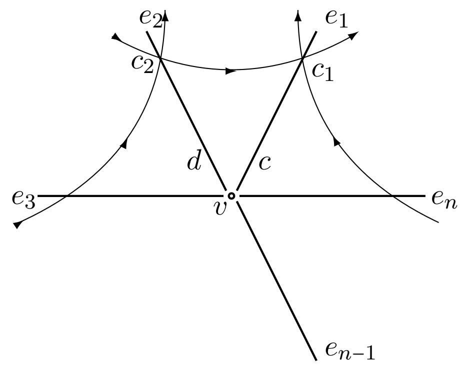


FIGURE 3. Walking around vertex v of graph $\mathbb{T}$

The d edges in $\mathbb{T}$ correspond to c edges in $\mathbb{T}^*$. Let E'_c be the set of edges of $\mathbb{T}^*$ coloured c, and let V'_c be the set of vertices of $\mathbb{T}^*$ adjacent to a c edge. Then as before we put $C' = (V'_c, E'_c)$, and again C' is not necessarily connected.

Theorem 2. *Each vertex of graph $\mathbb{T}$ has an even number of c edges adjacent to it (with loops being counted twice).*

Proof. Consider a vertex v of $\mathbb{T}$, with edges $e_1, e_2, \dots, e_n$ adjacent to it. This vertex v corresponds to a face f of D, and this face f is bounded by arcs of the link diagram with crossings $c_1, c_2, \dots, c_n$ corresponding to the edges $e_1, e_2, \dots, e_n$ of $\mathbb{T}$. Let us take a position on one of those arcs of D, between the edges e_n and e_1 of $\mathbb{T}$, and then walk around the face f until we return to our starting point. We note whether the direction we walk in agrees or disagrees with the orientation of D.

Without loss of generality we may assume that we begin walking in the direction compatible with the orientation of the link diagram D. We walk until we meet our first edge e_1. This edge has a label, either c or d. Let us assume that the edge e_1 is a c edge; see Figure 3. When we cross e_1 and continue walking onto the next strand we will walk against the direction of the orientation of D. In order for us to get back where we started and walk in the direction of the orientation of D, the arcs of D will have to change the direction in total an even number of times. Therefore there must be an even number of c edges at each vertex v of $\mathbb{T}$.

Now let us assume that the first edge we met, e_1, is a d edge. When we cross this edge the next arc of D has to be oriented in a direction compatible with the direction we are walking in, otherwise the edge e_1 would be a c edge. So we keep walking until we meet a c edge, and then the previous argument applies. If we do not encounter any c edges then all the edges adjacent to the vertex v are labelled d. □

Corollary 1. *Each connected component of C is Eulerian.*

Proof. This comes from the following equivalences: a graph G is Eulerian if an only if every vertex of G has an even degree, and if and only if the set of edges of G can be partitioned into cycles. (For a proof of these equivalences see [2], for example.) □

Corollary 2. *Each connected component of C' is Eulerian.*

Proof. This follows since C' is the Tait graph obtained from D using the other checkerboard colouring.

□

Corollary 3. *Any cycle of $\mathbb{T}$ contains an even number of d edges.*

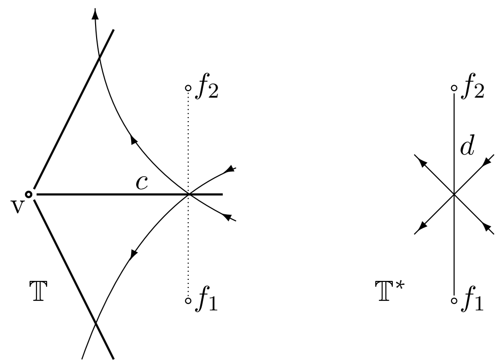


FIGURE 4. Relationship between the labels of edges of $\mathbb{T}$ and $\mathbb{T}^*$.

Proof. From Corollary 1 we can see that C can be partitioned into a set of cycles. $\mathbb{T}$ is a plane graph so each cycle of $\mathbb{T}$ separates the sphere into two regions (by the Jordan curve theorem). If either of these two regions contains no other edges of $\mathbb{T}$ then we call the cycle a *boundary*.

First we show that a boundary cycle of $\mathbb{T}$ contains an even number of d edges. Let C_1 be a boundary cycle of $\mathbb{T}$. We also have the dual Tait graph $\mathbb{T}^*$ corresponding to the same link diagram. The cycle C_1 in $\mathbb{T}$ corresponds to a star centered at vertex v^* in $\mathbb{T}^*$. Each edge e of C_1 has a corresponding edge e^* adjacent to v^*, with the opposite $\{c, d\}$ label; see Figure 4. Applying Theorem 2 to $\mathbb{T}^*$ we deduce that each vertex v^* has an even number of c edges adjacent to it. Hence the cycle C_1 of $\mathbb{T}$ contains an even number of d edges.

Now let C_2 be another boundary cycle, whose intersection with C_1 is a path. However many d edges there are in this path, if we were to delete the whole path the new face would have an even number of d edges. So a (non-boundary) cycle consisting of two faces must have an even number of d edges. Inductively, therefore, we see that any cycle contains an even number of d edges.

□

Note that, in fact, Theorem 2 and Corollaries 1, 2, and 3 are all equivalent to each other.

The underlying abstract graph $\boldsymbol{T}$ of $\mathbb{T}$ is equipped with an embedding $i : \boldsymbol{T} \to S^2$. This induces an embedding $\boldsymbol{T}^* \to S^2$, which we also denote i. Now take $i(C) \cup i(C')$, and denote the resulting graph by Φ. So Φ is formed by taking the standard immersion of $\mathbb{T} \cup \mathbb{T}^*$ and then setting

$$\Phi := (\mathbb{T} \cup \mathbb{T}^*) - (E(C^c) \cup E((C')^c)).$$

The construction of Φ and $\Phi^{\circledast}$ can be seen in Figures 5 and 6. Note that $\Phi^{\circledast}$ is not always a thickened tree. The graph $\Phi^{\circledast}$ will turn out to be the Seifert graph $\boldsymbol{S}$.

Theorem 3. *$\Phi^{\circledast}$ is the Seifert graph of the link diagram D corresponding to $\mathbb{T}$.*

Proof. We construct a mapping h from the vertex set $V(\boldsymbol{S})$ of the Seifert graph to the vertex set $V(\Phi^{\circledast})$. We draw on the same diagram the link diagram D together with the corresponding graphs $\mathbb{T}$ and $\mathbb{T}^*$, so that for each crossing of D there are two corresponding edges, one from $\mathbb{T}$ and one from $\mathbb{T}^*$. While resolving the crossings of the link diagram we remove the d edges of $\mathbb{T}$ and $\mathbb{T}^*$, so we have one edge for each crossing of D. Now we observe that the Seifert circles do not intersect any of the edges, and therefore given a Seifert circle we have a face of Φ, and hence a vertex of $\Phi^{\circledast}$ as required.

Suppose there are two Seifert circles in the same face of Φ. Each edge of this face crosses one of the dotted edges emerging from the Seifert circles, and we can label the edges 1 or 2 according to which Seifert circle they correspond to. There are just two vertices a and b in the boundary of the

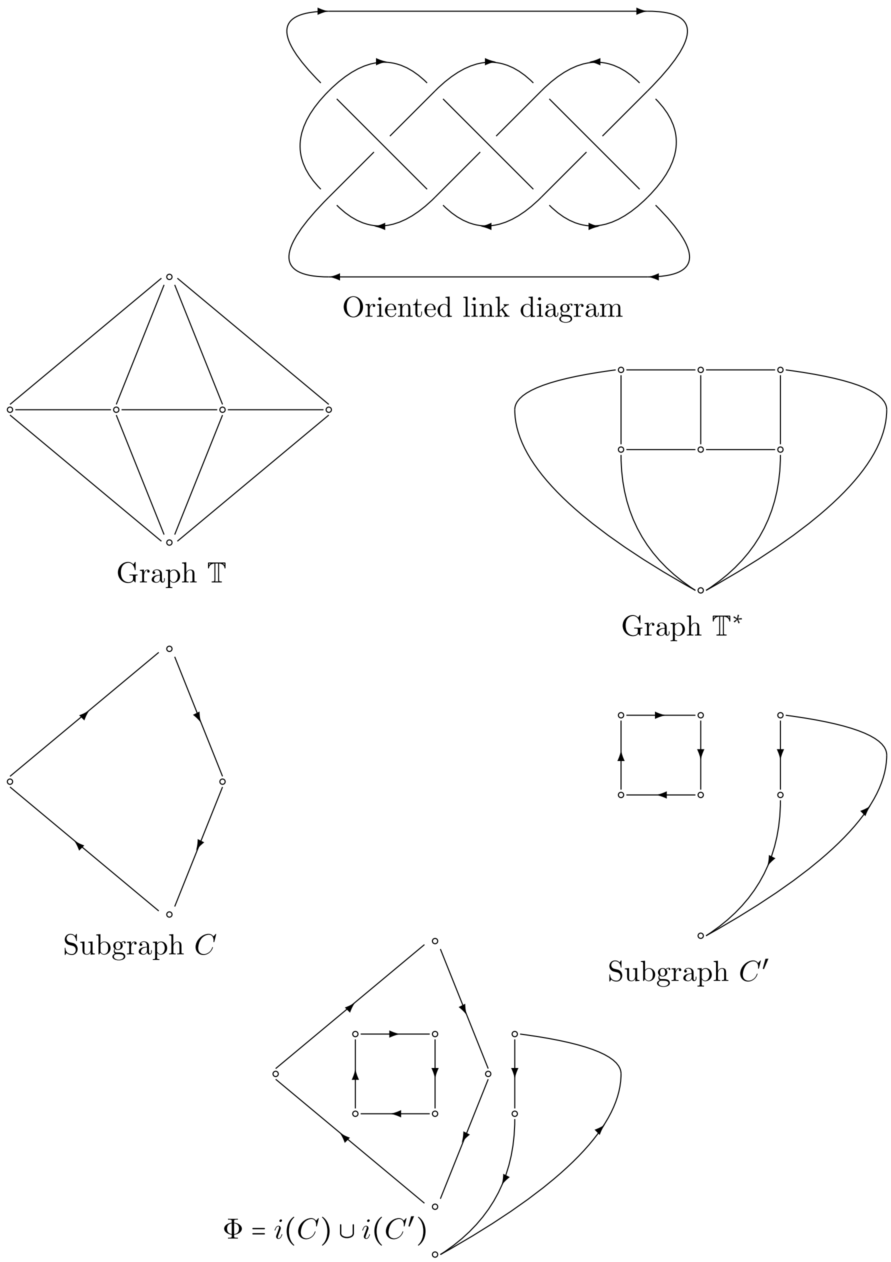


FIGURE 5. Construction of the graph Φ

face which have both 1 and 2 edges. These vertices are in the same face of U, so $a = b$, and hence the two circles must have been in different faces of Φ.

So h is injective.

We also know that if two vertices v and w in $\boldsymbol{S}$ are adjacent, then the corresponding Seifert circles are joined by a dotted edge, which implies that the faces in Φ share an edge and hence that $h(v)$ and $h(w)$ are adjacent in $\Phi^{\circledast}$.

This gives us a mapping $m : E(\boldsymbol{S}) \to E(\Phi^{\circledast})$, which must be a bijection, because any edge of $\Phi^{\circledast}$ comes ultimately from a specific crossing in D and hence from a specific edge of $\boldsymbol{S}$.

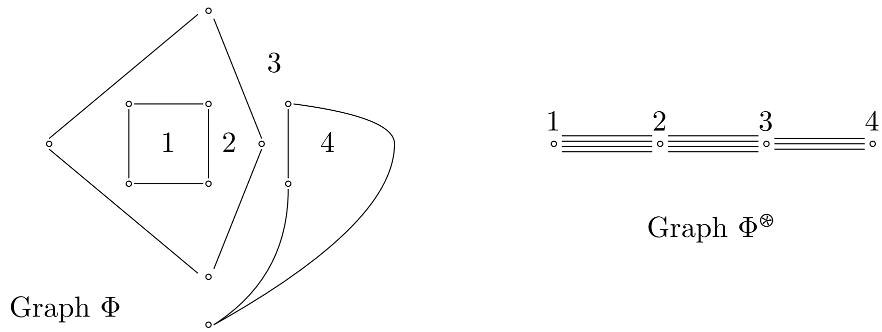


Figure 6. Construction of graph $\Phi^{\circledast}$

Finally, we show that h is a graph isomorphism. We already know that it is injective. Now let λ be in $V(\Phi^{\circledast})$. Choose a vertex μ adjacent to λ, denote $m^{-1}(\lambda\mu)$ by e, and suppose that $e = uv$, for $u, v \in V(\boldsymbol{S})$. Then $h(u)h(v) = \lambda\mu \in E(\Phi^{\circledast})$, which means that $\lambda = h(u)$ or $\lambda = h(v)$, and hence h is surjective. □

It is well known that Seifert graphs are bipartite. Here we see that this fact is a consequence of the Eulerian structures in Tait graphs.

Lemma 1. *$\Phi^{\circledast}$ is bipartite.*

Proof. Each component of the (disjoint) graphs C and C' is Eulerian, and so each vertex of Φ has even degree. Hence each face of $\Phi^{\circledast}$ has an even number of edges.

Any cycle in $\Phi^{\circledast}$ can be formed by adding face-cycles mod 2. Hence the result. □

Now we establish the conditions on a connected plane graph G which make it the Tait graph of a link diagram.

Theorem 4. *If, in a given $\{c, d\}$ colouring of the edges of the connected plane graph G, each component of C is Eulerian, and each component of C' is Eulerian, then G is the Tait graph of an oriented link diagram, the orientation coming from the $\{c, d\}$ colouring.*

Proof. Given our graph G we construct C, C', Φ, and $\Phi^{\circledast}$ as above. We first show that the faces of Φ must be either discs or annuli, by arguing that no two components of C' can lie in the same face of $i(C)$.

Consider a face f of $i(C)$, drawn in G. If it is a face of G, then it is a disc. If it is not a face of G, then it contains other edges and vertices of G, all the edges being marked d. These edges and vertices divide f into faces of G. But G is connected, so we can walk between any two of these faces via other such faces. Therefore, the part of C' lying in f must be connected, as required.

In $\Phi^{\circledast}$, this means that the deletion of any cut vertex splits $\Phi^{\circledast}$ into exactly two components. We also know, from Lemma 1, that $\Phi^{\circledast}$ is bipartite.

Choose a block B_1 of $\Phi^{\circledast}$. It is bipartite, and has no cut vertices. Choose an arbitrary transverse orientation of an edge $e^{\circledast}$ in B_1, and use this to determine a clockwise and anti-clockwise orientation at each of the ends of $e^{\circledast}$. Any other vertex in B_1 will be clockwise if it is an even distance from a clockwise vertex, and anti-clockwise otherwise. This is consistent because B_1 is bipartite, and so it has no odd cycles.

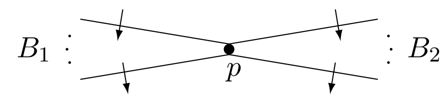
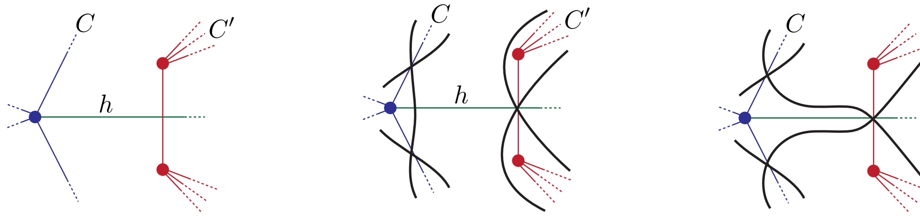


Figure 7. Orientation of neighbouring blocks



In $G \cup i(C')$. In the link D'. Obtaining D.

Figure 8. Splicing together arcs in the proof of Theorem 4.

Next, back in $\Phi^{\otimes}$, let p be a cut vertex of B_1. (If B_1 has no cut vertex, then $B_1 = \Phi^{\otimes}$ and we move to the next step.) By our observation above, there is a next block: call it B_2 and orient it as in Figure 7. Proceed in this way until all the edges of $\Phi^{\otimes}$ have been oriented. Finally, transfer this orientation back to the edges of Φ.

We will now describe how an oriented link diagram D, with the property that G is its Tait graph, can be recovered from the decorated graph Φ. On the plane embedding of G draw C and C' using the embedding i described above. (So C is a subgraph of G and C' is a subgraph of the standard embedding of G^*.) Form the link diagram D from this as follows. Place a crossing at the centre of each edge in C, and wherever an edge of C' intersects an edge of G. Orient each of these crossings so that the arcs are directed in the same direction as the edge of C or C' on which it lies. It remains to connect these crossings up in a way that is constant with their orientations. First connect all crossings that lie on C by following the faces of C, and all crossings that lie on C' by following the faces of C'. This can be done consistently because of the way that we assigned transverse orientations to the edges in each block of $\Phi^{\otimes}$. This gives an oriented link diagram D'. Finally, For each half-edge h of G with one end on a vertex v of $C \subset G$ and the other on an intersection point of G and C', splice together the arcs of the link diagram D' as indicated in Figure 8. This is consistant with the orientation of D' because of the way that the transverse orientation moved between the blocks of $\Phi^{\otimes}$. The resulting link diagram is D.

□

We can now prove our main result of this section, which was a characterization of Seifert graphs.

Proof of Theorem 1. Suppose that H is a plane graph and each component of $(H \cup H^*) - (A^c \cup A^*)$ is Eulerian. Then, by Theorem 4, it follows that H is a Tait graph of a link diagram D and that A is its set of c edges. By the definition of Φ and Theorem 3, it follows that G is the Seifert graph of D.

The converse follows immediately from Theorem 3. □

Remark 1. Theorem 3 provides a way to obtain the Seifert graph $\boldsymbol{S}(D)$ of a link diagram D. By Proposition 3, the Seifert graph can also be obtained as the underlying abstract graph of the partial

(a) A crossing in D and the corresponding edge in $\mathbb{T}$

(b) Crossings in D_2 and the corresponding edges in $\mathbb{T}(D_2)$

FIGURE 9. D, $T(D)$, D_2 and $T(D_2)$ locally at a crossing (shown without signs).

dual $\mathbb{T}_\sigma(D)^A$, where A is the set of all $(\pm,\pm)$ edges of $\mathbb{T}_\sigma(D)$. Thus we have

$$G^A \cong [(G \cup G^\star) - (A^c \cup A^\star)]^\circledast, \tag{1}$$

when G is a Tait graph and A is its set of c edges. It is natural to ask whether Equation (1) holds for any graph G and any $A \subseteq E(G)$. In [14], using a characterization of partial duals from [19], the first two authors show that Equation (1) does indeed hold for all G and A. This is significant, as it provides a way to construct partial dual graphs without having to compute the corresponding ribbon graphs. It also provides another illustration of the fruitful connection between knot theory and graph theory.

4. The graphs of r-fold parallels

This section is concerned with the structure of the (ribbon) graphs of r-fold parallels of link diagrams. Forming a parallel of a link is a fundamental and important operation. We begin by describing the operation on Tait graphs that corresponds to taking the r-fold parallel of a link diagram. This construction leads to an interesting sequence of Tait graphs. We then apply our results on Tait graphs to examine the ribbon graphs of r-fold parallels. In particular, we will determine the genus of the ribbon graph $\mathbb{G}(D_r)$ of an r-fold parallel of D in terms of the ribbon graph $\mathbb{G}(D)$ of D.

4.1. **Tait graphs of r-fold parallels.** Let D be a link diagram. Its *r-fold parallel*, D_r, is the diagram in which each link component of D is replaced by r copies, all parallel in the plane, each copy repeating the over- and under-crossing behaviour of the original link diagram. Figure 9 shows what happens to a crossing in the 2-fold parallel, together with the corresponding edges in the Tait graphs.

Let $\mathbb{T}(D)$ be the Tait graph of our link diagram D. Our aim is to determine the Tait graph $\mathbb{T}(D_r)$ of the r-fold parallel D_r of D directly from $\mathbb{T}(D)$. Thus we need to find a construction of the graph $\mathbb{T}_r(D)$ from $\mathbb{T}(D)$ that completes the commutative diagram:

$$\begin{array}{ccc} D & \longrightarrow & \mathbb{T}(D) \\ \downarrow & & \downarrow \\ D_r & \longrightarrow & \mathbb{T}(D_r) = \mathbb{T}_r(D) \end{array}. \tag{2}$$

In order to achieve this we will start with $\mathbb{T}(D)$, construct $\mathbb{T}_2(D)$, and then find a general recurrence relationship between $\mathbb{T}_{r+1}(D)$ and $\mathbb{T}_r(D)$.

For $\mathbb{T}_2(D)$ we need to define the overlay product of $\mathbb{T}$ and $\mathbb{T}^*$, denoted $\mathbb{T} \uplus \mathbb{T}^*$. In fact the overlay product is defined for any pair of plane graphs G and H that have the property that the vertices of each lie in the faces of the other, so we give the general definition.

Definition 5. Let G and H be abstract graphs, and suppose they are plane, and therefore equipped with embeddings $i : G \to S^2$ and $j : H \to S^2$. Suppose further that the vertices of each lie in the faces

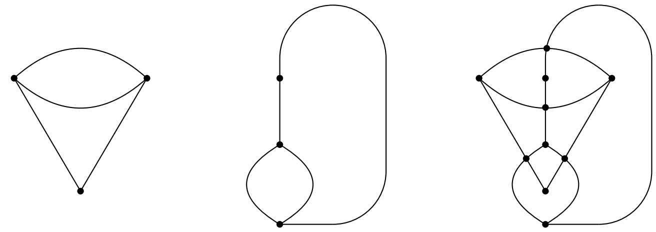


FIGURE 10. For the Figure Eight knot, the graphs $\mathbb{T}$, $\mathbb{T}^*$, and $\mathbb{T}_2 = \mathbb{T} \uplus \mathbb{T}^*$

of the other. The *overlay product*, $G \uplus H$, is the image $i(G) \cup j(H)$ with vertices added wherever an edge of G intersects an edge of H.

Every edge e in $G \uplus H$ arises from an edge e' in G or H. If e' has an edge weight, then e is assigned the same edge weight as e'.

In the case of the plane graph G and its dual G^*, the embedding j is induced from i, and the condition that the vertices of each lie in the faces of the other is satisfied by construction.

Definition 6. Let $\mathbb{T}$ be a Tait graph, then we define $\mathbb{T}_r$ inductively by $\mathbb{T}_1 := \mathbb{T}$ and

$$\mathbb{T}_r := \mathbb{T}^*_{r-1} \uplus \mathbb{T}.$$

We note that $\mathbb{T}_r$ can be defined in this way since, at each step, the vertices of $\mathbb{T}^*_r$ are the faces of $\mathbb{T}_r$, which are subsets of the faces of $\mathbb{T}$, and the vertices of $\mathbb{T}$ are also vertices of $\mathbb{T}_r$, and hence lie in faces of $\mathbb{T}^*_r$.

The following theorem shows that the overlay product is the construction required to complete the commutative diagram (2).

Theorem 5. *Let D be a link diagram, then $\mathbb{T}_r(D) = \mathbb{T}_1(D_r)$.*

Proof. We use induction on r. The base case, $r = 2$, follows easily by considering a single crossing, as in Figure 9.

Suppose that $\mathbb{T}_{r-1}(D) = \mathbb{T}_1(D_{r-1})$. Then $\mathbb{T}_r(D) = (\mathbb{T}_{r-1}(D))^* \uplus \mathbb{T}_1 = (\mathbb{T}_1(D_{r-1}))^* \uplus \mathbb{T}_1$. We need to show that

$$(\mathbb{T}_1(D_{r-1}))^* \uplus \mathbb{T}_1 = \mathbb{T}_1(D_r). \tag{3}$$

Consider a crossing in D. In $\mathbb{T}_1(D_{r-1})$ it yields an $r \times r$ chess board, with the long diagonal black. In $\mathbb{T}_1(D_r)$ there is a new row and column, so the chess board now has size $(r+1) \times (r+1)$. The long diagonal is still black, however. The pattern of black squares in the $(r+1) \times (r+1)$ board can be obtained by taking the pattern of white squares in the $r \times r$ board, making them black, and then inserting a new long diagonal. In terms of our graphs, this means taking the dual of $\mathbb{T}_1(D_{r-1})$, and then the overlay product with $\mathbb{T}_1(D)$ inserts the new long diagonal, giving Equation (3). □

The following two results show that the structure of the graphs $\mathbb{T}_r(D)$ is quite tightly constrained.

Lemma 2. *Let D be a link diagram. When r is even all the faces of $\mathbb{T}_r(D)$ are squares. In the case when r is odd, if f is a face of $\mathbb{T}(D)$ and f_r is the subgraph of $\mathbb{T}_r(D)$ that is contained in f, then f_r contains a copy of f and all other faces of $\mathbb{T}_r(D)$ are squares.*

Proof. The statements are true for $r = 1$ and $r = 2$. We use induction on r. Suppose that the statements are true for all r less than k. Consider a face of $\mathbb{T}_{k+1}$. It is either a face of $\mathbb{T}^*_k$, or a part of a face of $\mathbb{T}^*_k$. In the case where a face of $\mathbb{T}_{k+1}$ is a face of $\mathbb{T}^*_k$, we can immediately go back one more step. Each face of $\mathbb{T}^*_k$ corresponds to a vertex of $\mathbb{T}_k$, which must be either (a) a face of $\mathbb{T}^*_{k-1}$; (b) a new vertex of degree four formed by the overlay; or (c) a vertex of $\mathbb{T}$. In cases (a) and (b) we can deduce that the face of $\mathbb{T}^*_{k+1}$ must either be a face of $\mathbb{T}^*_{k-1}$ or a square. Case (c) does not arise, because if the face of $\mathbb{T}^*_k$ had come from a vertex of $\mathbb{T}$ then it would have been subdivided.

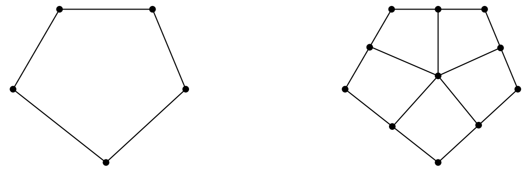


FIGURE 11. A face of $\mathbb{T}$, and the corresponding faces of $\mathbb{T} \uplus \mathbb{T}^*$

This leads us to the second case where a face of $\mathbb{T}_{k+1}$ is a part of a face of $\mathbb{T}_k^*$. A face of $\mathbb{T}_k^*$ may be subdivided in two ways. It either has a vertex of $\mathbb{T}$ in it, in which case it is subdivided into squares, or it has an edge of $\mathbb{T}$ crossing it. Here, too, it is subdivided into squares. Hence the result. □

The overlay product construction, together with Euler's equation, lead to the following formulae for the number of edges, faces and vertices of $T_r(D)$ which will be of use later.

Lemma 3. *Let D be a link diagram, f, e, and v be the number of faces, edges and vertices of graph $\mathbb{T}(D)$, respectively, and let f_r, e_r, and v_r be the number of faces, edges and vertices of $\mathbb{T}_r(D)$, respectively. For r even we have:*

$$e_r = r^2 e, \qquad f_r = \frac{r^2 e}{2}, \qquad v_r = 2 + \frac{r^2 e}{2};$$

and for r odd:

$$e_r = r^2 e, \qquad f_r = f + \frac{e(r^2-1)}{2}, \qquad v_r = v + \frac{e(r^2-1)}{2}.$$

Proof. If the original link diagram D had e crossings then the r-fold parallel of D will have $r^2 e$ crossings. Since the number of edges of $\mathbb{T}_r$ is equal to the number of crossings of the link D_r, we have $e_r = r^2 e$.

If r is even all the faces are squares, and so $4f_r = 2e_r$, and hence $f_r = \frac{r^2 e}{2}$. Now Euler's equation gives $v_r = 2 + \frac{r^2 e}{2}$.

If r is odd we count the faces in more detail. Let $f_{r,n}$ be the number of faces in $\mathbb{T}_r$ bounded by n edges. Then, by Lemma 2,

$$f_r = x + f_{1,1} + f_{1,2} + f_{1,3} + f_{1,4} + f_{1,5} + \cdots = x + f,$$

for some x. Now let us count the edges.

$$2e_r = f_{r,1} + 2f_{r,2} + 3f_{r,3} + 4f_{r,4} + 5f_{r,5} + \cdots = 4x + f_{1,1} + 2f_{1,2} + 3f_{1,3} + 4f_{1,4} + 5f_{1,5} + \cdots = 4x + 2e.$$

Therefore $x = \frac{e(r^2-1)}{2}$, and so $f_r = f + \frac{e(r^2-1)}{2}$. Finally, we use Euler's equation again to deduce that $v_r = v + \frac{e(r^2-1)}{2}$, as required. □

We conclude this section with some notation and a technical result that we require later. Given any plane graph G with r-fold overlay G_r, there is a natural map

$$\varphi_r : E(G_r) \to E({G_{r-1}}^*) \cup E(G)$$

induced by the definition of the overlay product: the edge $e \in E(G_r)$ lies in an edge $\varphi_r(e)$ of ${G_{r-1}}^*$ or G. We call the map φ_r the *projection* of $E(G_r)$ onto $E({G_{r-1}}^*) \cup E(G)$.

When the link diagram D is not alternating, its Tait graph $\mathbb{T}(D)$ will be signed. Evidently, given an edge $e \in \mathbb{T}(D)$, the corresponding edge $e^* \in \mathbb{T}(D)^*$ will have the opposite sign. The sign of an edge $e \in E(G_r)$ is the same as the sign of $\varphi_r(e)$, whether this is in $E({G_{r-1}}^*)$ or $E(G)$. Of course, if $\varphi_r(e) \in E({G_{r-1}}^*)$ then the sign of e is opposite to the sign of $\varphi_r(e)^* \in E(G_{r-1})$. The following lemma tells us that if we take two adjacent edges in G_r with the property that one edge arises from an edge in ${G_{r-1}}^*$ and the other from an edge in G, then the two edges are of different signs.

Lemma 4. *Let G be a signed plane graph, G_r denote its r-fold overlay, and φ_r be the projection of $E(G_r)$ onto $E(G_{r-1}{}^*) \cup E(G)$. If $e \in \varphi_r^{-1}(E(G_{r-1}{}^*))$ and $f \in \varphi_r^{-1}(E(G))$ are adjacent, they have different signs.*

Proof. Because $\varphi_r(e) \in E(G_{r-1}{}^*)$ we must have $\varphi_r(e)^* \in E(G_{r-1})$. Now $\varphi_{r-1}(\varphi_r(e)^*) \in E(G)$, because e and f are adjacent in G_r. In fact, $\varphi_{r-1}(\varphi_r(e)^*) = \varphi_r(f)$. Therefore $\varphi_r(e)^*$ has the same sign as f, which implies that $\varphi_r(e)$ has the opposite sign to f, and hence the result. □

4.2. **The genus of the ribbon graphs of r-fold parallels.** In this subsection we determine how the genus of a ribbon graph of a link diagram relates to the genus of the ribbon graphs of its parallels.

Let D be a link diagram and D_r be its r-fold parallel. In forming the parallel, every crossing c of D gives rise to a set, $C_r(c)$, of crossings of D_r. If s is a state of D then we can form a state s_r of D_r by, for each crossing c, splicing all of the crossings in D_r in the same way as c was spliced in the state s. So if a crossing c of D is A-spliced (respectively B-spliced) in the state s, then every crossing of in $C_r(c)$ is A-spliced (respectively B-spliced) in the state s_r of D_r.

Theorem 6. *Let D be a link diagram and s be a state of D. Then*

$$g(\mathbb{G}(D_{r+1}, s_{r+1})) = (r+1)\cdot g(\mathbb{G}(D,s)) + r^2 \cdot e(\mathbb{G}(D,s)) - r.$$

The proof of this theorem will follow from Theorem 7 below.

Theorem 6 has immediate applications to the special ribbon graphs of a link diagram described above.

Corollary 4. *Let D be a link diagram and D_{r+1} be its $(r+1)$-fold parallel. Then*

(1) $g(\mathbb{A}(D_{r+1})) = (r+1)\cdot g(\mathbb{A}(D)) + r^2 \cdot e(\mathbb{A}(D)) - r$;
(2) $g(\mathbb{B}(D_{r+1})) = (r+1)\cdot g(\mathbb{B}(D)) + r^2 \cdot e(\mathbb{B}(D)) - r$;
(3) $g(\mathbb{S}(D_{r+1})) = (r+1)\cdot g(\mathbb{S}(D)) + r^2 \cdot e(\mathbb{S}(D)) - r$.

Proof. For the first item, suppose s is obtained by choosing an A-splicing at each crossing. Then s_{r+1} is also obtained by choosing an A-splicing at each crossing. We have $\mathbb{G}(D,s) = \mathbb{A}(D)$ and $\mathbb{G}(D_{r+1}, s_{r+1}) = \mathbb{A}(D_{r+1})$, and the result follows from Theorem 6. The second item is proved similarly.

As for the third item, note that if c is a crossing in D with oriented sign + (respectively −) then every crossing in $C_r(c)$ also has oriented sign + (respectively −). Thus if s is a state of D such that $\mathbb{G}(D,s) = \mathbb{S}(D)$, it follows that s_{r+1} is a state of D_{r+1} such that $\mathbb{G}(D_{r+1}, s_{r+1}) = \mathbb{S}(D_{r+1})$. The result then follows from Theorem 6. □

The Turaev genus, $g_t(L)$, (see [1, 16, 21]) of a link L is defined as the minimum genus $g(\mathbb{A}(D))$, over all link diagrams D of L. The following corollary provides a bound on the Turaev genus of an r-fold parallel of a link in terms of the Turaev genus of the original link.

Corollary 5. *Let L be a link and L_r be its r-fold parallel. Then an upper bound on the Turaev genus, $g_t(L_r)$, of L_r is*

$$g_t(L_r) \le (r+1)\cdot g_t(L) + r^2 c - r,$$

where c is the crossing number of any diagram D for L attaining the Turaev genus.

Proof. The corollary follows immediately from the definition of the Turaev genus and Corollary 4. □

Theorem 6 also provides a way to calculate the genus of a ribbon graph of D_r in terms of the Tait graph of D.

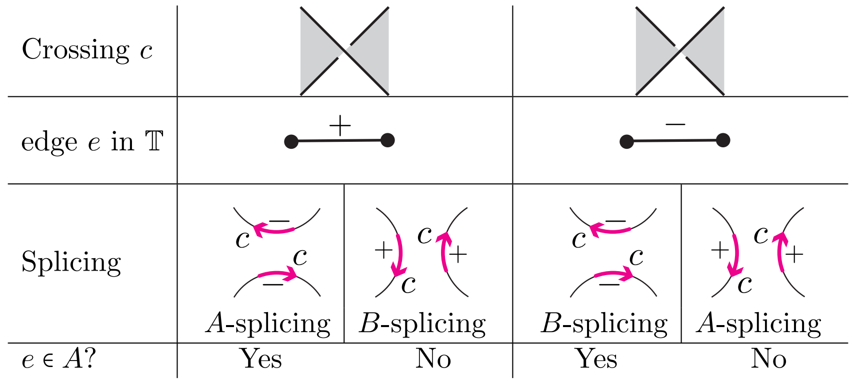


FIGURE 12. The correspondence between splicings in D and edges in $\mathbb{T}(D)$

Corollary 6. *Let D be a link diagram, $\mathbb{T} = \mathbb{T}(D)$ be its Tait graph, and s be a state of D. Then if A is the set of edges of $\mathbb{T}$ such that $\mathbb{T}^A = \mathbb{G}(D, s)$,*

$$2g(\mathbb{G}(D_{r+1}, s_{r+1})) = (2r^2 + r + 1) \cdot e(\mathbb{T}) - (r+1) \cdot (p(\mathbb{T} - A^c) - p(\mathbb{T} - A)) + 2.$$

Proof. By Theorem 6,

$$g(\mathbb{G}(D_{r+1}, s_{r+1})) = (r+1) \cdot g(\mathbb{G}(D, s)) + r^2 \cdot e(\mathbb{A}(D)) - r.$$

Since $\mathbb{G}(D, s) = \mathbb{T}^A$, Proposition 1 gives

$$g(\mathbb{G}(D, s)) = 1 + e(\mathbb{T})/2 - p(\mathbb{T} - A^c)/2 - p(\mathbb{T} - A)/2,$$

and so

$$g(\mathbb{G}(D_{r+1}, s_{r+1})) = (2r^2 + r + 1)/2 \cdot e(\mathbb{T}) - (r+1) \cdot p(\mathbb{T} - A^c)/2 - (r+1) \cdot p(\mathbb{T} - A)/2 + 1.$$

□

We will now turn our attention to the proof of Theorem 6. In fact, we will prove the following more general result which expresses $g(\mathbb{A}(D_{r+1}))$ in terms of the genus of lower order parallels of D. Theorem 6 follows by repeated application of this result.

Theorem 7. *Let D be a link diagram and s be a state of D. Then for each $r \in \mathbb{N}$ we have*

$$g(\mathbb{G}(D_{r+1}), s_{r+1}) = g(\mathbb{G}(D_r, s_r)) + g(\mathbb{G}(D, s)) + r \cdot e(\mathbb{G}(D, s)) - 1.$$

To prove the theorem we begin by discussing some relations of $\mathbb{G}(D_{r+1}, s_{r+1})$ with partial duals. Suppose that D is a link diagram, s is a state of D, and $\mathbb{T}(D)$ is its Tait graph. Since the ribbon graphs of a link diagram are all partial duals of the Tait graph, by Proposition 2, we know that there exists a unique set of edges $A \subseteq E(\mathbb{T}(D))$, such that $\mathbb{G}(D, s) = \mathbb{T}(D)^A$ (the uniqueness uses the fact that the edges are signed), and similarly, for each $r \in \mathbb{N}$, there is a subset of edges $A_r \subseteq E(\mathbb{G}(D_r, s_r))$ such that $\mathbb{G}(D_r, s_r) = \mathbb{T}(D_r)^{A_r} = (\mathbb{T}_r(D))^{A_r}$. Our first aim is to find a description of the set A_r in terms of the set A.

In the state s_r of D_r, every crossing in $C_r(c)$ (which is the set of crossings in D_r arising from the crossing c in D) is spliced in the same way as the crossing c of D. Now let e be the edge of $\mathbb{T}(D)$ corresponding to c, and $E_r(e)$ be the set of edges of D_r corresponding to the crossings in $C_r(c)$. Then from Figure 12, we see that e is or is not in A depending on the splicing at c. Now, let e_r be an edge in $E_r(e)$, and $c_r \in C_r(c)$ be the crossing corresponding to e_r. Then c_r and c are spliced in the same way in the states s_r and s respectively. So to determine if $e_r \in A_r$, which is the set of edges of $\mathbb{T}(D_r)$ that describe the state s_r, we consult Figure 12 and see that $e_r \in A_r$ if and only if

- e and e_r have the same sign and $e \in A$; or
- e and e_r have different signs and $e \notin A$.

Thus the set of edges A_r such that $\mathbb{G}(D_r, s_r) = \mathbb{T}(D_r)^{A_r}$ is constructed by, for each edge e_r of $\mathbb{T}(D_r)$, looking at the edge e in $\mathbb{T}(D)$ that is associated with e_r, and putting e_r in A_r if e and e_r have the same sign and $e \in A$, or if e and e_r have different signs and $e \notin A$. So,

$$(4) \quad A_r = \{e \in E(\mathbb{T}(D)_r) \mid [\rho(e) \in A \text{ with same sign as } e] \text{ or } [\rho(e) \notin A \text{ with different sign to } e]\},$$

where $\rho(e)$ is the edge in $\mathbb{T}(D)$ such that $e \in E_r(\rho(e))$.

This discussion in summarized by the following lemma.

Lemma 5. *Let D be a link diagram and s be a state of D. If $\mathbb{G}(D,s) = \mathbb{T}(D)^A$, then $\mathbb{G}(D_r, s_r) = \mathbb{T}(D_r)^{A_r}$, where A_r is given by Equation* (4).

We will also need the following technical lemma.

Lemma 6. *Let $\mathbb{T}$ be a Tait graph and, for each $k \in \mathbb{N}$, A_k be given by Equation* (4). *Then*

(1) $p(\mathbb{T}_{r+1} - A_{r+1}) = p({\mathbb{T}_r}^* - {A_r}^*) + p(\mathbb{T} - A)$;
(2) $p(\mathbb{T}_{r+1} - (A_{r+1})^c) = p({\mathbb{T}_r}^* - ({A_r}^*)^c) + p(\mathbb{T} - A^c)$.

Proof. For the first item, let φ be the projection of $E(G_{r+1})$ onto $E({G_r}^*) \cup E(G)$, as described at the end of Subsection 4.1. In addition, let H denote the plane subgraph of $G_{r+1} = {G_r}^* \uplus G$ induced by $\varphi^{-1}(E(G))$, and let K denote the plane subgraph of $G_{r+1} = {G_r}^* \uplus G$ induced by $\varphi^{-1}(E({G_r}^*))$. Then $H \cup K = G_{r+1}$ and $H \cap K$ is the set of vertices created by the overlay ${G_r}^* \uplus G$.

Let $v \in H \cap K$. Then v is incident to exactly four edges. Call these edges e, e', f, f', two of which are in H and two of which are in K. Suppose that e and e' are in H, and that f and f' are in K.

By Lemma 4, e and e' are both of the same sign, m say, and f and f' are both of the same sign $-m$. It then follows from the definition of A_{r+1}, that either $e, e' \in A_{r+1}$, or $f, f' \in A_{r+1}$. Consequently no connected component of $\mathbb{T}_{r+1} - A_{r+1}$ has edges in both H and K. So every connected component of $\mathbb{T}_{r+1} - A_{r+1}$ is a connected component of exactly one of the plane graphs $H - A_{r+1}$ and $K - A_{r+1}$. Therefore

$$p(\mathbb{T}_{r+1} - A_{r+1}) = p(H - A_{r+1}) + p(K - A_{r+1}).$$

Finally, by the definition of the overlay product, we have that $p(H - A_{r+1}) = p(\mathbb{T} - A)$ and that $p(K - A_{r+1}) = p({\mathbb{T}_r}^* - {A_r}^*)$, and the result follows.

The proof of the second item is similar. □

We can now prove Theorem 7

Proof of Theorem 7. Let $\mathbb{T} := \mathbb{T}(D)$ and $\mathbb{T}_k := \mathbb{T}(D_k)$ be the Tait graphs of D and D_k respectively. In addition let s be a state of D and $A \subseteq E(\mathbb{T})$ be such that $\mathbb{G}(D,s) = \mathbb{T}^A$. Then by Theorem 5, $\mathbb{T}(D_k) = \mathbb{T}_k$, the k-fold overlay of $\mathbb{T}$. Moreover, by Lemma 5, $\mathbb{G}(D_k, s_k) = {\mathbb{T}_k}^{A_k}$, where A_k is given by Equation (4). We then have

$$2g(\mathbb{G}(D_{r+1}, s_{r+1})) = 2g({\mathbb{T}_{r+1}}^{A_{r+1}}) = 2 + e({\mathbb{T}_{r+1}}^{A_{r+1}}) - p(\mathbb{T}_{r+1} - {A_{r+1}}^c) - p(\mathbb{T}_{r+1} - A_{r+1}),$$

where the second equality follows from Proposition 1. Using Lemma 6 to expand the above expression, we have

$$2g(\mathbb{G}(D_{r+1}, s_{r+1})) = 2 + e({\mathbb{T}_{r+1}}^{A_{r+1}}) - p({\mathbb{T}_r}^* - ({A_r}^*)^c) - p(\mathbb{T} - A^c) - p({\mathbb{T}_r}^* - {A_r}^*) - p(\mathbb{T} - A).$$

Now, since partial duality does not change the number of edges, we can use Lemma 3 twice to get

$$e({\mathbb{T}_{r+1}}^{A_{r+1}}) = e(\mathbb{T}_{r+1}) = (r+1)^2 e(\mathbb{T}) = (2r+1)e(\mathbb{T}) + e(\mathbb{T}_r) = (2r+1)e(\mathbb{T}) + e({\mathbb{T}_r}^*).$$

Thus we can write

$$2g(\mathbb{G}(D_{r+1}, s_{r+1})) = [2r \cdot e(\mathbb{T}) - 2] + [2 + e({\mathbb{T}_r}^*) - p({\mathbb{T}_r}^* - ({A_r}^*)^c) - p({\mathbb{T}_r}^* - {A_r}^*)] \\ + [2 + e(\mathbb{T}) - p(\mathbb{T} - A^c) - p(\mathbb{T} - A)] .$$

Which, by Proposition 1 gives

$$\begin{aligned} 2g(\mathbb{G}(D_{r+1}, s_{r+1})) &= 2g(({\mathbb{T}_r}^*)^{{A_r}^*}) + 2g(\mathbb{T}^A) + 2r \cdot e(\mathbb{T}) - 2 \\ &= 2g((\mathbb{T}_r)^{A_r}) + 2g(\mathbb{T}^A) + 2r \cdot e(\mathbb{T}) - 2 \\ &= 2g(\mathbb{G}(D_r, s_r)) + 2g(\mathbb{G}(D, s)) + 2r \cdot e(\mathbb{G}(D, s)) - 2, \end{aligned}$$

as required. □

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