

# Euler Characteristics and Chromatic Polynomials

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## Abstract

Given a graph we show how to construct a family of manifolds whose Euler characteristics are the values of the chromatic polynomial of the graph at various integers. The manifolds are simple generalisations of configuration spaces.

## 1 Introduction

One of the defining features of the chromatic polynomial  $c(G, \lambda)$  of a graph  $G$  is that it satisfies the *deletion-contraction* relation: if  $e$  is an edge of  $G$  joining vertices  $v$  and  $w$ , let  $G \setminus e$  be the graph  $G$  with  $e$  deleted, and let  $G/e$  be the graph  $G \setminus e$  with vertices  $v$  and  $w$  identified. Then:

$$c(G, \lambda) = c(G \setminus e, \lambda) - c(G/e, \lambda),$$

simply because in any proper colouring of the vertices of  $G \setminus e$ ,  $v$  and  $w$  either receive the same colour (in which case the colouring is also proper for  $G/e$ ) or different colours (in which case the colouring is proper for  $G$ ).

There is a long exact sequence in singular homology theory which, in certain circumstances, mimics the deletion-contraction relation. It is the *Leray sequence* [11], which can be described as follows. Let  $Y$  be a closed submanifold (of codimension  $m$ ) of the manifold  $X$ , and suppose that  $Y$  has orientable normal bundle. Then the Leray sequence is

$$\dots \rightarrow H_i(X - Y) \rightarrow H_i(X) \rightarrow H_{i-m}(Y) \rightarrow H_{i-1}(X - Y) \rightarrow \dots$$

Now choose  $Y$  to be the diagonal  $\Delta$  in the space  $X = M \times M$ , where  $M$  is an  $m$ -dimensional orientable manifold (noting that the normal bundle of a diagonal may be identified with the tangent bundle and is, therefore, orientable). In  $M \times M - \Delta$  the two variables have to avoid each other, in  $M \times M$  they may coincide, and in  $\Delta$  they must coincide. Our main result shows how to use this

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to express  $c(G, \lambda)$  in terms of the Euler characteristic of a family of spaces associated with  $G$ .

There are a number of constructions in algebraic geometry or algebraic topology that are designed to extract chromatic information from a graph. In [1] the cycle matroid of the graph is first embedded in projective space, and then a sequence of blow-ups applied, obtaining an algebraic variety from which the chromatic polynomial of the original graph can be derived. Slightly closer to our own approach is that in [5], where colourings of planar graphs are obtained from residues of 1-forms on a degenerate algebraic curve called the graph curve.

In [2] complexes of graph homomorphisms are used to establish results on the chromatic number such as the Lovász conjecture: we will discuss this in more detail in section 5. Finally, given a link diagram [10] constructs a chain complex of graded vector spaces whose graded Euler characteristic is the Kauffman bracket polynomial. A similar construction can be applied to graphs, and we explore this in section 6, showing how it differs from our work.

Here is a brief plan of the article. In section 2 we show how to define a generalised configuration space corresponding to a graph, and we also demonstrate in detail how the Leray sequence then makes the Euler characteristics obey a deletion-contraction relation. This does not quite establish our main result, for we still have to show that in the case when the graph consists just of one vertex we really are counting its colourings: this we do in section 3. Section 4 discusses some (elementary) graph-theoretic results which can now be established from properties of the Euler characteristic. Then, as we have noted, in sections 5 and 6 we explore the relationship between our work and other approaches. (Finally, an appendix is used to keep some technicalities out of the way.)

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## 2 Generalised Configuration Spaces

Given a graph  $G$  with vertex set  $V = \{v_1, \dots, v_n\}$  and edge set  $E$  we construct a manifold  $M_G$  (of dimension  $mn$ ) as follows. Let  $M$  be an  $m$ -dimensional manifold (to be specified later). In the product space  $M^{\times n}$  each copy of  $M$  is associated to a particular vertex in  $G$ . Each edge  $e = v_i v_j$  in  $G$  defines a “diagonal” subspace  $\Delta_e$  of  $M^{\times n}$  by

$$\Delta_e = \{(x_1, \dots, x_n) \in M^{\times n} : x_i = x_j\}.$$

Now

$$M_G = M^{\times n} - \cup_{e \in E} \Delta_e.$$

In the special case when  $G$  is a complete graph the space  $M_G$  is exactly the *configuration space*  $\mathbb{F}_n(M)$  of [7]; hence the title of this section.

We apply the Leray sequence to one edge, say  $f$ , at a time. Let

$$X - Y = M_G, \quad X = M^{\times n} - \cup_{e \in E, e \neq f} \Delta_e, \quad Y = X \cap \Delta_f,$$

so that  $M_{G \setminus f} = X$  and  $M_{G/f} = Y$ . Then the complete long exact sequence is

$$0 \rightarrow H_{mn}(M_G) \rightarrow H_{mn}(M_{G \setminus f}) \rightarrow H_{mn-m}(M_{G/f}) \rightarrow H_{mn-1}(M_G) \rightarrow \cdots \rightarrow 0.$$

If it could be arranged that the connecting homomorphisms were zero, then this sequence would split into short exact sequences of groups whose ranks would then satisfy a deletion-contraction relation. This is tempting for another reason: these connecting homomorphisms are dual to the residue map in cohomology, and the construction in [5], while different, also imposed conditions on residues of differential forms. However, these connecting homomorphisms are quite subtle, and we can avoid having to know anything about them by considering the Euler characteristics of these three spaces.

The alternating sum of the Betti numbers in this sequence is zero:

$$\beta_{mn}(M_G) - \beta_{mn}(M_{G \setminus f}) + \beta_{mn-m}(M_{G/f}) - \beta_{mn-1}(M_G) + \cdots = 0,$$

and hence, for even  $m$  as we now suppose, the Euler characteristics satisfy

$$\chi(M_G) - \chi(M_{G \setminus f}) + \chi(M_{G/f}) = 0.$$

This is our deletion-contraction relation.

### 3 Choosing the manifold $M$

The chromatic polynomial is a rather special case of the Tutte polynomial. In fact,

$$c(G, \lambda) = (-1)^{r(G)} \lambda^{k(G)} T(G; 1 - \lambda, 0).$$

Here  $n(G)$  is the number of vertices of  $G$ , the number of components is  $k(G)$ , and its rank is  $r(G) = n(G) - k(G)$ .

The Tutte polynomial is in quite a strong sense universal, as the following shows (see [4]). Here a multigraph is a graph that may have loops and multiple edges,  $m(G)$  is the number of edges of  $G$ , and  $\nu(G) = m(G) - r(G)$ .

**Theorem 1** (see [4]) *Let  $\mathcal{G}$  be the set of finite multigraphs, and let  $E_n$  be the edgeless graph on  $n$  vertices. Then there is a unique map  $U : \mathcal{G} \rightarrow \mathbb{Z}[\xi, \eta, \alpha, \sigma, \tau]$  such that for any  $n \geq 1$*

$$U(E_n) = \alpha^n$$

and for any edge  $e$  of  $G$

$$U(G) = \begin{array}{ll} \xi U(G/e) & e \text{ a bridge} \\ \eta U(G \setminus e) & e \text{ a loop} \\ \sigma U(G \setminus e) + \tau U(G/e) & \text{otherwise.} \end{array}$$

In addition:

$$U(G) = \alpha^{k(G)} \sigma^{\nu(G)} \tau^{r(G)} T(G; \xi/\tau, \eta/\sigma).$$

Now we define our “atomic” space  $M$  in such a way that when  $G$  consists of just one vertex then  $\chi(M)$  is the number of ways of colouring  $G$  with  $\lambda$  colours. We choose  $M = \mathbf{CP}^{\lambda-1}$ . Then  $\chi(M) = \lambda$ .

**Theorem 2** *Let  $G$  be a graph and  $M_G$  the generalised configuration space defined above, with  $M = \mathbf{CP}^{\lambda-1}$ . Then*

$$c(G, \lambda) = \chi(M_G).$$

**Proof** It is immediate from the behaviour of the Euler characteristic on products that  $\chi(E_n) = \lambda^n$ . Now, by Theorem 1, we have our result.  $\square$

It would be tempting to choose  $M = \widetilde{\mathbf{CP}}^1$ , which is  $\mathbf{CP}^1$  with  $\lambda + 2$  points blown up. Then  $\chi(M) = -\lambda$ , and an application of Theorem 1 would give

$$\chi(M_G) = (-1)^{k(G)} c(G, \lambda).$$

A possible advantage of this approach would be that the parameter  $\lambda$  appears in the individual Betti numbers. However, this choice of  $M$  fails, because the blow-up takes place in the underlying real manifold  $S^2$ , and a blow-up of a real even-dimensional manifold at a point is not orientable.

## 4 Graph-theoretic results

We start by showing how the iteration of the Leray sequences can be used to establish the formula

$$c(G, \lambda) = \sum_{S \subseteq E(G)} (-1)^{|S|} \lambda^{k(G:S)}$$

where  $G : S$  is the subgraph of  $G$  given by  $V(G : S) = V(G)$  and  $E(G : S) = S$ . We note first that  $\lambda^{k(G:S)} = c(G/S, \lambda)$ , where the graph  $G/S$  is obtained from  $G : S$  by contracting each edge in  $S$ .

We apply the formula

$$\chi(M_G) = \chi(M_{G \setminus f}) - \chi(M_{G/f})$$

to any edge  $f \in E(G)$ . Then choose an edge  $g$ , say, in  $G \setminus f$  and another one  $h$  in  $G/f$ , giving

$$\chi(M_{G \setminus f}) = \chi(M_{G \setminus f \setminus g}) - \chi(M_{(G \setminus f)/g})$$

and

$$\chi(M_{G/f}) = \chi(M_{G/f \setminus h}) - \chi(M_{G/f/h}).$$

We continue with this, taking the edges of  $G$  in any order, until no edges are left. The original  $\chi(M_G)$  will then have been expressed in terms of  $\chi(M_{G/S})$  for all possible subsets  $S \subseteq E(G)$ . But  $\chi(M_{G/S}) = c(G/S, \lambda) = \lambda^{k(G:S)}$ , so we have an expression for  $c(G, \lambda)$  as a signed sum of  $\lambda^{k(G:S)}$ . Finally, we note that the coefficient of  $\lambda^{k(G:S)}$  is  $(-1)^{|S|}$  because each contraction introduces a minus sign.  $\square$

The next result is also elementary, but is more interesting for us: our proof is a topological one, using the notion of a fibration.

Let  $K_p$  denote the complete graph on  $p$  vertices. Suppose  $G = G_1 \cup G_2$ , and  $G_1 \cap G_2 = K_p$ . Then the chromatic polynomials satisfy

$$c(G, \lambda) = \frac{c(G_1, \lambda)c(G_2, \lambda)}{c(K_p, \lambda)}.$$

In fact a more general formula is true: all we need is that  $G_1 \cap G_2 = H$ , where those vertices in  $H$  which are joined to vertices in  $G - H$  form a complete subgraph of  $H$ . Then the chromatic polynomials satisfy

$$c(G, \lambda) = \frac{c(G_1, \lambda)c(G_2, \lambda)}{c(H, \lambda)}.$$

We can demonstrate the corresponding relation between the Euler characteristics as follows. For any fibration

$$\begin{array}{ccc} F & \hookrightarrow & E \\ & & \downarrow \\ & & B \end{array}$$

we have  $\chi(E) = \chi(F)\chi(B)$ . In general, given a subgraph  $H$  of  $G$ , the space  $M_G$  will not be a fibration over  $M_H$ . However, if as we have assumed those vertices in  $H$  which are joined to vertices in  $G - H$  form a complete subgraph of  $H$ , then  $M_G$  is a fibration over  $M_H$ . (See the appendix for a proof of this claim.)

So, with  $G = G_1 \cup G_2$ , and  $G_1 \cap G_2 = H$ , we have the fibration

$$\begin{array}{ccc} F & \hookrightarrow & M_G \\ & & \downarrow \\ & & M_H \end{array}$$

for some space  $F$ , and hence

$$\chi(M_G) = \chi(F)\chi(M_H). \tag{1}$$

We also have the fibrations

$$\begin{array}{ccc} F_1 & \hookrightarrow & M_G \\ & & \downarrow \\ & & M_{G_2} \end{array}$$

and

$$\begin{array}{ccc} F_2 & \hookrightarrow & M_G \\ & & \downarrow \\ & & M_{G_1} \end{array}$$

from which we obtain

$$\chi(M_G) = \chi(F_1)\chi(M_{G_2}) = \chi(F_2)\chi(M_{G_1}). \tag{2}$$

From (1) and (2) and the fact that  $F = F_1 \times F_2$  we immediately deduce

$$\chi(M_G) = \frac{\chi(M_{G_1})\chi(M_{G_2})}{\chi(M_H)},$$

as required.

## 5 Graph homomorphisms

Given graphs  $G$  and  $H$  a *homomorphism*  $\phi : G \rightarrow H$  is defined to be a mapping  $\phi : V(G) \rightarrow V(H)$  such that for all pairs  $v_1, v_2 \in V(G)$  with  $v_1v_2 \in E(G)$  we have  $\phi(v_1)\phi(v_2) \in E(H)$  (see for example chapter six in [9]). As we indicated in the introduction, these homomorphisms form complexes, which have recently been used to establish the Lovász conjecture [2]. Here we will show how to interpret the homomorphisms from the point of view of our manifolds  $M_G$ . It is plausible that this approach will yield more than just the chromatic polynomial of  $G$ .

We will define an induced mapping  $\phi^* : M_H \rightarrow M_G$ . A point  $z \in M_H$  may be thought of as a mapping  $z : V(H) \rightarrow M$  such that if  $w_1w_2 \in E(H)$  then  $z(w_1) \neq z(w_2)$ . The mapping  $\phi^*(z) : V(G) \rightarrow M$  is defined by setting  $\phi^*(z)(v) = z(\phi(v))$ , and if  $v_1v_2 \in E(G)$  then  $\phi(v_1)\phi(v_2) \in E(H)$ , whence  $z(\phi(v_1)) \neq z(\phi(v_2))$  and hence  $\phi^*(z)(v_1) \neq \phi^*(z)(v_2)$  as required.

There are three special cases of interest to us.

1. If  $H$  has a complete subgraph  $K_n$  there is a homomorphism  $\phi : K_n \rightarrow H$ , and the induced mapping  $\phi^* : M_H \rightarrow M_{K_n}$  is a fibration, as we saw in the previous section.
2. Suppose  $G$  and  $H$  are identical except that  $H$  has exactly one more edge than  $G$ . Then the identity mapping on  $V(G) = V(H)$  is the elementary homomorphism  $\iota$  which simply inserts an edge. The induced mapping  $\iota^* : M_H \rightarrow M_G$  arises from a mapping of the form  $M^{\times 2} \setminus \Delta \rightarrow M^{\times 2}$ , which in turn induces the corresponding map in the Leray sequence.
3. Suppose  $G$  and  $H$  are identical except that two vertices in  $G$  not joined by an edge have been identified in  $H$ . Then the identification mapping is the elementary homomorphism  $\kappa$ , which simply contracts a single non-edge. The induced mapping  $\kappa^* : M_H \rightarrow M_G$  arises from  $M \rightarrow \Delta \subset M^{\times 2}$ , and will be used in the next section to define a chain mapping.

A homomorphism from  $G$  to  $K_n$  is a colouring of  $G$  with  $n$  colours. So the number of induced mappings  $M_{K_n} \hookrightarrow M_G$  is the chromatic polynomial  $c(G, n)$ . This can be seen directly, by deletion-contraction. Let  $e = v_1v_2 \in E(G)$ . Any induced mapping  $\phi^* : M_{K_n} \rightarrow M_{G \setminus e}$  either has  $\phi(v_1) = \phi(v_2)$  or  $\phi(v_1) \neq \phi(v_2)$ . If the former,  $\phi^*$  is an induced mapping into  $M_{G/e}$ , and if the latter,  $\phi^*$  is an induced mapping into  $M_G$ . Note that this argument does not require  $M$  to be such that  $\chi(M) = \lambda$ .

Given a configuration space one expects a braid group to be lurking: here it is. Consider a colouring  $\phi : G \rightarrow K_n$ . Then  $\phi^* : M_{K_n} \hookrightarrow M_G$ . Here, as we have noted,  $M_{K_n}$  is precisely a configuration space, and so we have another induced map on the fundamental groups, giving  $(\phi^*)_* : P_n \rightarrow \pi_1(M_G)$ , where  $P_n$  is the pure braid group on  $n$  strings associated to  $M$ , and  $\pi_1(M_G)$  can be thought of as a pure braid group in which some pairs of strings (corresponding to the non-edges of  $G$ ) are allowed to pass through each other. When  $M$  is the 2-sphere  $P_n$  only differs from Artin's classical pure braid group by having one extra relation: see [8].

## 6 Comparison with the Khovanov homology of link diagrams

If we switch from homology to sheaf cohomology we can relax the conditions (that  $M$  be even-dimensional, and on the orientability of the normal bundle). Retaining those conditions for a moment we have by Poincaré duality that

$$\chi(M_G) = \sum_i (-1)^i \beta_c^i(M_G),$$

but in any case we have

$$c(G, \lambda) = \sum_i (-1)^i \beta_c^i(M_G). \quad (3)$$

This is proved exactly as before: the long exact sequence we use instead of the Leray sequence (and in fact Poincaré dual to it) is obtained as follows. Let  $k_{M_G}$  be the “skyscraper sheaf” on  $M^{\times n}$  with support on  $M_G$ , so that  $H_c^i(M_G) = H_c^i(M^{\times n}; k_{M_G})$ . Given any edge  $f$  of  $G$  there is a short exact sequence of sheaves on  $M^{\times n}$ :

$$0 \rightarrow k_{M_G} \rightarrow k_{M_G \setminus f} \rightarrow k_{M_G/f} \rightarrow 0.$$

Apply  $\Gamma_c(M^{\times n}; *)$  to this sequence to obtain the required long exact sequence.

Finally, note that now we can even choose  $M$  to be a set consisting of  $\lambda$  points, in which case the right hand side of equation (3) becomes the number of global sections of  $k_{M_G}$ , and so (3) is a tautology! This choice of  $M$  goes against the spirit of this article, but it does allow an interesting comparison with the Khovanov homology of link diagrams [10].

Our construction is parallel to that in [3], but it applies to graphs rather than link projections.  $K$  is a functor from the category of graphs to the category of chain complexes, defined recursively as follows:–

$$K(\emptyset) = 0 \rightarrow \mathbf{Z} \rightarrow 0,$$

and

$$K(\bullet \sqcup G) = V \otimes K(G),$$

where  $V$  is a vector space of dimension  $\lambda$ . Specifically, let us take  $V = C_0(M)$  where  $M$  is the space consisting of  $\lambda$  points. The remaining ingredient in our definition arises from deletion-contraction:–

$$K(G) = \mathcal{F}(0 \rightarrow K(G/e) \xrightarrow{d} K(G \setminus e) \rightarrow 0),$$

where  $\mathcal{F}$  is the operation of taking direct sums along the diagonals of a double complex to obtain a single complex. The mapping  $d$  is induced by the mapping  $\kappa^*$  of the previous section. The chromatic polynomial of  $G$  is now obtained by taking the Euler characteristic of the corresponding chain complex:–

$$c(G, \lambda) = \chi(K(G)). \quad (4)$$

Though the details are clearly different, this  $K$  resembles the Khovanov functor. In particular, the Jones polynomial of a link is essentially the Euler characteristic of the Khovanov functor applied to that link and (4) provides counterpart to this statement for the chromatic polynomial.

## 7 Appendix

### 7.1

The *Leray sequence* [11] can be derived as follows. Let  $Y$  be a closed submanifold (of codimension  $m$ ) of the manifold  $X$ , and suppose that  $Y$  has orientable normal bundle. Then in the homology sequence of the pair  $(X, X - Y)$

$$\dots \rightarrow H_i(X - Y) \rightarrow H_i(X) \rightarrow H_i(X, X - Y) \rightarrow H_{i-1}(X - Y) \rightarrow \dots$$

we may use excision and the Thom isomorphism to replace  $H_i(X, X - Y)$  by  $H_{i-m}(Y)$ . We obtain the Leray sequence

$$\dots \rightarrow H_i(X - Y) \rightarrow H_i(X) \rightarrow H_{i-m}(Y) \rightarrow H_{i-1}(X - Y) \rightarrow \dots$$

The connecting homomorphism  $H_{i-m}(Y) \rightarrow H_{i-1}(X - Y)$  replaces each point of a cycle by a small normal  $m - 1$  sphere.

### 7.2

**Theorem 3** *Suppose that  $K$  is a complete subgraph of  $G$ . Then the projection  $\pi : M_G \rightarrow M_K$  is a locally trivial fibration.*

**Proof** This is adapted from the proof of the first theorem in [6], the only difference being that here  $G$  is not necessarily a complete graph, so that only  $M_K$  has to be a configuration space ( $\mathbb{F}_n(M)$ ). This has the effect of making the fibres of the projection  $\pi$  more complicated.

Suppose  $K$  has  $r$  vertices, and let  $\mathbf{b} = (b_1, \dots, b_r)$  be a point in  $M_K$ . Choose neighbourhoods  $U_1, \dots, U_r$  in  $M$  such that  $\overline{U}_1, \dots, \overline{U}_r$  are mutually disjoint and  $b_i \in U_i$ ,  $i = 1, \dots, r$ .

For each  $i = 1, \dots, r$  and for each  $x \in U_i$  there is a homeomorphism  $\gamma_i(x)$  from  $M$  to itself satisfying:

- (i)  $\gamma_i(x)(x) = b_i$ , and
- (ii)  $\gamma_i(x)(y) = y, \forall y \in M \setminus U_i$ .

Denote by  $F_{\mathbf{b}}$  the fibre of  $\pi$  above  $\mathbf{b}$ , and let  $U = U_1 \times \dots \times U_r$ . Then we have the local trivialization

$$\begin{array}{ccc} U \times F_{\mathbf{b}} & \xrightarrow{\phi_U} & \pi^{-1}(U) \\ \downarrow & & \downarrow \\ U & \xrightarrow{\cong} & U, \end{array}$$

where

$$\phi_U(x_1, \dots, x_r, y) = (x_1, \dots, x_r, (\gamma_1^{-1}(x_1) \circ \dots \circ \gamma_r^{-1}(x_r))(y)).$$

## References

- [1] P. Aluffi, A blow-up construction and graph colouring, *Discrete Math.* **145** (1995), 11–35.
- [2] E. Babson and D. Kozlov, Proof of the Lovász conjecture, arXiv:math.CO/0402395.
- [3] D. Bar-Natan, On Khovanov’s categorification of the Jones polynomial, *Algebr. Geom. Topol.* **2** (2002), 337–370.
- [4] B. Bollobás and O. Riordan, A Tutte polynomial for coloured graphs, *Combin. Probab. Comput.* **8** (1999), 45–93.
- [5] C. Ciliberto and R. Miranda, Graph curves, colourings, and matroids, in: *Zero-Dimensional Schemes* (eds. F. Orecchia and L. Chiantini), Walter de Gruyter 1994, pp. 89–111.
- [6] E. Fadell and S. Husseini, *Geometry and Topology of Configuration Spaces*, Springer-Verlag 2001.
- [7] E. Fadell and L. Neuwirth, Configuration spaces, *Math. Scand.* **10** (1962), 111–118.
- [8] E. Fadell and J. Van Buskirk, The braid groups of  $E^2$  and  $S^2$ , *Duke Math. J.* **29** (1962), 243–258.
- [9] C. Godsil and G. Royle, *Algebraic Graph Theory*, Springer-Verlag 2001.
- [10] M. Khovanov, A categorification of the Jones polynomial, *Duke Math. Jour.* **101** (2000), 359–426.
- [11] J. Leray, Le calcul différentiel et intégral sur un variété analytique complexe, *Bull. Soc. Math. France* **87** (1959), 81–180.