The Elements of Twistor Theory

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1 Introduction

These are notes from my lecture at the *Twistor String Theory* workshop held at the Mathematical Institute Oxford, 10th–14th January 2005. They are almost the same as the slides I used, except that I have compressed them to save paper and corrected a couple of errors.

2 Twistor geometry

Twistor space $\mathbb{T}$ is $\mathbb{C}^4$ with coordinates

$$Z^\alpha = (Z^0, Z^1, Z^2, Z^3).$$

Projective twistor space $\mathbb{PT}$ is $\mathbb{CP}^3$. $Z \in \mathbb{PT}$ has homogeneous coordinates $[Z^0, Z^1, Z^2, Z^3]$.

A plane in $\mathbb{PT}$ is a $\mathbb{CP}^2$ given by an equation of the form

$$Z^\alpha A_\alpha = 0.$$

In other words it is determined by a dual (projective) twistor $[A_0, A_1, A_2, A_3]$.

A line in $\mathbb{PT}$ is a $\mathbb{CP}^1$ given by the intersection of two planes

$$Z^\alpha A_\alpha = Z^\beta B_\beta = 0.$$

Of course there is some freedom in the choice of A_α and B_β.

What is the space of these lines in $\mathbb{PT}$?

Each is determined by a skew simple (0,2) twistor $L_{\alpha\beta}$. The condition for simplicity can be written

$$3L_{\alpha[\beta}L_{\gamma\delta]} = L_{\alpha\beta}L_{\gamma\delta} + L_{\alpha\gamma}L_{\delta\beta} + L_{\alpha\delta}L_{\beta\gamma} = 0,$$

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which defines a quadric Q in $\mathbb{CP}^5$ (called the Klein quadric).

By changing the coordinates we can see that Q is actually the space of generators of the cone

$$T^2 + V^2 - W^2 - X^2 - Y^2 - Z^2 = 0$$

in $\mathbb{C}^6$.

Here is the change of coordinates: $T = \frac{i}{\sqrt{2}}(L_{03} - L_{12})$
$V = L_{23} + \frac{1}{2}L_{01}$
$W = L_{23} - \frac{1}{2}L_{01}$
$X = \frac{i}{\sqrt{2}}(L_{02} - L_{13})$
$Y = \frac{-1}{\sqrt{2}}(L_{02} + L_{13})$
$Z = \frac{-i}{\sqrt{2}}(L_{12} + L_{03})$
Here is the embedding of $\mathbb{M}$ in the cone:

$$x^a \to (x^0, \frac{1}{2}(1 - x^b x_b), -\frac{1}{2}(1 + x^b x_b), x^1, x^2, x^3)$$

Thought of in $\mathbb{R}^6$ our cone is the $O(2,4)$ null cone of

$$ds^2 = dT^2 + dV^2 - dW^2 - dX^2 - dY^2 - dZ^2.$$

Each of its generators (except those for which $W - V = 0$) meets the plane

$$W - V = 1$$

in a point, and the intersection of this plane and the cone is just Minkowski space $\mathbb{M}$.

So the space of generators is a compactification $\mathbb{M}^c$ of $\mathbb{M}$.

It is the conformal compactification: the extra generators form a null cone at infinity.

We have shown that there is a four real dimensional family of lines in $\mathbb{PT}$ corresponding to $\mathbb{M}$, but we have not so far shown how to identify them in $\mathbb{PT}$.

For a twistor Z to lie on a line L it must satisfy two linear equations. Except when the line is given by $Z^2 = Z^3 = 0$, these can be written

$$\begin{pmatrix} Z^0 \\ Z^1 \end{pmatrix} = \frac{i}{\sqrt{2}} \begin{pmatrix} x^0 + x^3 & x^1 + ix^2 \\ x^1 - ix^2 & x^0 - x^3 \end{pmatrix} \begin{pmatrix} Z^2 \\ Z^3 \end{pmatrix}$$

where x^a is the space-time point corresponding to L. More concisely, if we write $Z^\alpha = (\omega^A, \pi_{A'})$ we have

$$\omega^A = ix^{AA'}\pi_{A'}.$$

If Z also lies on the line corresponding to $y^{AA'}$, then

$$x^{AA'}\pi_{A'} = y^{AA'}\pi_{A'}$$

and so the matrix $x^{AA'} - y^{AA'}$ must be singular. The condition for this is that x^a and y^a are null-separated.

If in

$$\omega^A = ix^{AA'}\pi_{A'}$$

we think of the twistor as fixed and solve for the point $x^{AA'}$ we find that

$$x^{AA'} = x_0^{AA'} + \mu^A\pi^{A'}$$

for arbitrary μ^A.

So $Z^\alpha = (\omega^A, \pi_{A'})$ corresponds to this alpha-plane: it is a totally null two complex dimensional plane in complex Minkowski space.

$\mathbb{PT}$	$\mathbb{CM}^c$
complex projective line	point
point	alpha-plane
intersection of lines	null-separation of points

In general an alpha-plane will have no real point, but when it does it contains a whole real null ray: if $x_0^{AA'}$ is real then so is

$$x^{AA'} = x_0^{AA'} + r\bar{\pi}^A\pi^{A'}$$

for any real r.

If Z^α is the twistor for this alpha-plane then

$$\begin{aligned}
\Sigma(Z) &= \omega^A\bar{\pi}_A + \bar{\omega}^{A'}\pi_{A'} \\
&= Z^0\bar{Z}^2 + Z^1\bar{Z}^3 + \bar{Z}^0 Z^2 + \bar{Z}^1 Z^3 \\
&= 0.
\end{aligned}$$

This Hermitian form Σ divides $\mathbb{PT}$ into three regions:

$$\Sigma(Z) > 0 \qquad \mathbb{PT}^+$$

$$\Sigma(Z) = 0 \qquad \mathbb{PN}$$

$$\Sigma(Z) < 0 \qquad \mathbb{PT}^-$$

$\mathbb{PN}$ is the space of real null rays: it is a five real dimensional manifold with a C-R structure. We could imagine discovering projective twistor space this way.

If $x^{AA'}$ is real then any Z lying on the corresponding line in $\mathbb{PT}$ satisfies

$$\omega^A = ix^{AA'}\pi_{A'}$$

and hence has $\Sigma(Z) = 0$.

Thus points in real (compactified) Minkowski space correspond to lines lying entirely in $\mathbb{PN}$.

Any two twistors on a given line in $\mathbb{PN}$ represent null rays through the corresponding point in $\mathbb{M}$.

So intrinsically the line in $\mathbb{PN}$ is the celestial sphere of the space-time point.

Lines lying entirely in $\mathbb{PT}^+$ correspond to points

$$z^{AA'} = x^{AA'} - iy^{AA'}$$

with $y^{AA'}$ timelike and future-pointing, or in other words points $z^{AA'}$ in the future tube.

This will lead later to a very elegant twistor description of positive frequency, using the fact that positive frequency fields can be characterized by having holomorphic extensions into the future tube.

3 The Penrose transform

Consider the twistor function

$$f(Z^\alpha) = \frac{1}{(A_\alpha Z^\alpha)(B_\beta Z^\beta)}$$

where $A_\alpha = (A_A, A^{A'}), B_\alpha = (B_A, B^{A'})$ are two constant dual twistors.

We aim to calculate a field at the point $x^{AA'}$ in $\mathbb{CM}$. Restrict Z^α to the line L_x so that

$$\begin{aligned} A_\alpha Z^\alpha &= (iA_A x^{AA'} + A^{A'})\pi_{A'} \equiv \alpha^{A'}\pi_{A'} \\ B_\alpha Z^\alpha &= (iB_A x^{AA'} + B^{A'})\pi_{A'} \equiv \beta^{A'}\pi_{A'} \end{aligned}$$

and consider the contour integral

$$\varphi(x) = \frac{1}{2\pi i} \oint \frac{1}{(\alpha^{A'}\pi_{A'})(\beta^{B'}\pi_{B'})} \pi_{C'} d\pi^{C'}.$$

This is well defined on $\mathbb{CP}^1$ since the integrand has total homogeneity zero. Further, there will exist a contour around which to do the integral provided the two poles are distinct:

$$\alpha^{A'}\beta_{A'} \neq 0.$$

This is satisfied unless L_x intersects the line $A_{[\alpha}B_{\beta]}$, or in other words unless x is on the null cone of a point y in $\mathbb{CM}$.

The contour integral above yields

$$\varphi(x) = \frac{2}{A_A B^A (x-y)^2},$$

which is a solution of the wave equation

$$\Box \varphi = 0.$$

Solutions of the massless free field equations for spin $n/2$

$$\nabla^{AA'}\varphi_{A' \ldots B'} = 0$$

(for positive frequency positive helicity) and

$$\nabla^{AA'}\psi_{A \ldots B} = 0$$

(for positive frequency negative helicity) may be solved in a similar way.

Let $f_{-n-2}(Z^\alpha)$ and $f_{n-2}(Z^\alpha)$ be holomorphic and homogenous of degrees $-n-2$ and $n-2$.

Define fields $\varphi_{A' \ldots B'}(x)$ and $\psi_{A \ldots B}(x)$ by:

$$\varphi_{A' \ldots B'}(x) = \frac{1}{2\pi i} \oint \pi_{A'} \ldots \pi_{B'} f_{-n-2}(Z^\alpha) \pi_{C'} d\pi^{C'}$$

$$\psi_{A \ldots B}(x) = \frac{1}{2\pi i} \oint \frac{\partial}{\partial \omega^A} \cdots \frac{\partial}{\partial \omega^B} f_{n-2}(Z^\alpha) \pi_{C'} d\pi^{C'}$$

where both integrands are first restricted so that Z lies on L_x.

The remarkable thing here is that φ and ψ are automatically solutions of the massless free field equations.

It is easy to write down examples of such functions by choosing rational functions, and we can easily check when their singularities are such that the contour integral is well defined.

In general this can be fiddly, though. Also, it is clear that many twistor functions will give the same space-time field.

Define

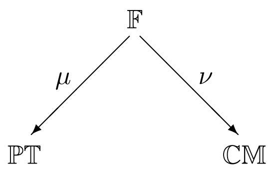
$$\begin{aligned} F_1 &= \{L_1 : L_1 \text{ is a one-dimensional} \\ &\quad\ \text{subspace of } \mathbb{T}\} \\ F_2 &= \{L_2 : L_2 \text{ is a two-dimensional} \\ &\quad\ \text{subspace of } \mathbb{T}\} \\ \mathbb{F} = F_{1,2} &= \{(L_1, L_2) : L_1 \text{ and } L_2 \text{ as above} \\ &\quad\ \text{with } L_1 \text{ a subspace of } L_2\}. \end{aligned}$$

Then

$$F_1 = \mathbb{PT}, \quad F_2 = \mathbb{CM}^c$$

and $\mathbb{F}$ can be identified with the primed spin bundle, with (local) coordinates $(x^a, \pi_{A'})$.

We have the double fibration



where
$\nu : (x^a, \pi_{A'}) \to x^a$
$\mu : (x^a, \pi_{A'}) \to (ix^{AA'}\pi_{A'}, \pi_{A'})$
The inverse image of a point in $\mathbb{PT}$ under μ is the whole alpha-plane in $\mathbb{F}$. So a function $f(x^a, \pi_{A'})$ on $\mathbb{F}$ pushes down to a function on $\mathbb{PT}$ if it is constant on alpha-planes:

$$\pi^{A'} \nabla_{AA'} f = 0.$$

We may restrict the double fibration to the future tube $\mathbb{CM}^+$ and corresponding regions $\mathbb{F}^+$ and $\mathbb{PT}^+$.

Let $\mathcal{U}$ be a cover of $\mathbb{F}^+$ by open sets U_i.

An element of $C^0(\mathcal{U}, \mathcal{S})$ is a function f_i (of type $\mathcal{S}$, defined on U_i) for each i.

An element of $C^1(\mathcal{U}, \mathcal{S})$ is a function f_{ij} (of type $\mathcal{S}$, defined on $U_i \cap U_j$) for each i, j.

We have maps

$$\begin{array}{rcl} \delta_0 : C^0 & \to & C^1 \\ \{f_i\} & \to & \{\rho_i f_j - \rho_j f_i\} \end{array}$$

and

$$\begin{array}{rcl} \delta_1 : C^1 & \to & C^2 \\ \{f_{ij}\} & \to & \{\rho_i f_{jk} + \rho_j f_{ki} + \rho_k f_{ij}\} \end{array}$$

In our example we had just two sets:

$$U_0 = \{Z^\alpha : Z^\alpha A_\alpha \neq 0\},\, U_1 = \{Z^\alpha : Z^\alpha B_\alpha \neq 0\}$$

and our function was actually f_{01} defined on $U_0 \cap U_1$.

Let $\mathcal{Z}'_n(m)$ be the sheaf of germs of symmetric n-index primed spinor fields $\varphi_{A'\ldots B'}(x,\pi)$ which are

holomorphic on $\mathbb{F}^+$,
homogeneous of degree m in $\pi_{A'}$,
solutions of

$$\nabla_A{}^{A'} \varphi_{A'\ldots B'} = 0.$$

Consider the onto map

$$\begin{array}{rcl} \pi^{C'} : \mathcal{Z}'_{n+1}(m-1) & \to & \mathcal{Z}'_n(m) \\ \varphi_{A'\ldots B'C'} & \to & \varphi_{A'\ldots B'C'}\pi^{C'}. \end{array}$$

We have the short exact sequence

$$0 \to \mathcal{T} \xrightarrow{i} \mathcal{Z}'_{n+1}(-1) \xrightarrow{\pi^{C'}} \mathcal{Z}'_n(0) \to 0$$

where $\mathcal{T}$ is just the kernel of the map $\pi^{C'}$.

$$\psi_{A'\ldots B'C'} \in \mathcal{T} \quad \Rightarrow \quad \psi_{A'\ldots B'C'}\pi^{C'} = 0,$$

and hence

$$\psi_{A'\ldots B'C'} = \pi_{A'} \ldots \pi_{B'}\pi_{C'} f(x,\pi).$$

Further, the zero rest mass equation on $\psi_{A'\ldots B'C'}$ implies that $f(x,\pi)$ must satisfy

$$\pi^{A'} \nabla_{AA'} f = 0.$$

It is therefore a twistor function homogeneous of degree $-n-2$.

We may now write our short exact sequence as

$$0 \to \mathcal{T}(-n-2) \overset{\pi_{A'}\dots\pi_{B'}\pi_{C'}}{\to} \mathcal{Z}'_{n+1}(-1) \overset{\pi^{C'}}{\to} \mathcal{Z}'_n(0) \to 0.$$

In the following array the squares commute and the rows are exact, but in the columns we only have $\mathrm{Im}\delta_0 \subseteq \mathrm{Ker}\delta_1$, in general. However, in the central column $\mathrm{Im}\delta_0 = \mathrm{Ker}\delta_1$.

$$\begin{array}{cccccccc}
0 \to & C^0(\mathcal{U},\mathcal{T}) & \to & C^0(\mathcal{U},\mathcal{Z}'_{n+1}(-1)) & \to & C^0(\mathcal{U},\mathcal{Z}'_n(0)) & \to 0 \\
 & & & h_{iA'\dots B'C'} & & h_{iA'\dots B'C'}\pi^{A'} & \\
 & \downarrow \delta_0 & & \downarrow \delta_0 & & \downarrow \delta_0 & \\
0 \to & C^1(\mathcal{U},\mathcal{T}) & \to & C^1(\mathcal{U},\mathcal{Z}'_{n+1}(-1)) & \to & C^1(\mathcal{U},\mathcal{Z}'_n(0)) & \to 0 \\
 & f_{ij} & & \pi_{A'}\dots\pi_{B'}\pi_{C'}f_{ij} & & & \\
 & \downarrow \delta_1 & & \downarrow \delta_1 & & \downarrow \delta_1 &
\end{array}$$

$$\begin{aligned}
h_{iA'\dots B'C'} &= \frac{1}{2\pi i}\oint_{\gamma_i}\frac{\lambda_{A'}\dots\lambda_{B'}\lambda_{C'}f_{ij}\lambda_{D'}d\lambda^{D'}}{\pi_{E'}\lambda^{E'}} \\
h_{iA'\dots B'C'}\pi^{A'} &= \rho_i\varphi_{A'\dots B'}
\end{aligned}$$

We started with f_{ij} in the kernel of δ_1, but for any f_{ij} in the image of δ_0 we would have obtained zero, so we actually want the class of f_{ij} in

$$\mathrm{Ker}\delta_1/\mathrm{Im}\delta_0 = H^1(\mathcal{U},\mathcal{T}(-n-2)).$$

To see that this procedure leads to an isomorphism we proceed as follows.

From our short exact sequence there is a corresponding long exact sequence of cohomology groups, which includes this part:

$$\begin{aligned}
\cdots &\to \check{H}^0(\mathbb{F}^+;\mathcal{Z}'_{n+1}(-1)) \\
\to \check{H}^0(\mathbb{F}^+;\mathcal{Z}'_n(0)) &\to \check{H}^1(\mathbb{F}^+;\mathcal{T}(-n-2)) \to \\
\check{H}^1(\mathbb{F}^+;\mathcal{Z}'_{n+1}(-1)) &\to \cdots
\end{aligned}$$

The first term is global sections of $\mathcal{Z}'_{n+1}(-1)$ over $\mathbb{F}^+$.

For fixed x^a, such a section would give a global section homogeneous of degree -1 on $\mathbb{CP}^1$, and there are no such sections.

Thus $\check{H}^0(\mathbb{F}^+;\mathcal{Z}'_{n+1}(-1)) = 0$.

A similar argument, slightly more technical, shows that

$$\check{H}^1(\mathbb{F}^+;\mathcal{Z}'_{n+1}(-1)) = 0$$

so that we are left with an isomorphism

$$\check{H}^0(\mathbb{F}^+; \mathcal{Z}'_n(0)) \overset{\delta^*}{\cong} \check{H}^1(\mathbb{F}^+; \mathcal{T}(-n-2))$$

The term on the left represents solutions of the zero rest mass equations on $\mathbb{F}^+$, homogeneous of degree zero in $\pi_{A'}$. They must therefore be independent of $\pi_{A'}$, and defined on $\mathbb{CM}^+$.

The term on the right is a cohomology group on $\mathbb{F}^+$ but the coefficients are *twistor* functions, so

$$\check{H}^1(\mathbb{F}^+; \mathcal{T}(-n-2)) \cong \check{H}^1(\mathbb{PT}^+; \mathcal{O}(-n-2)).$$

We have shown that the space of zero rest mass fields $\varphi_{A'\ldots B'}(x)$ of helicity n, holomorphic on $\mathbb{CM}^+$, is isomorphic to $\check{H}^1(\mathbb{PT}^+; \mathcal{O}(-n-2))$.

In fact this works for other open subsets

$$X \subset \mathbb{CM}^c$$

and the corresponding spaces

$$Y = \nu^{-1}(X) \subset \mathbb{F}$$

and

$$Z = \mu(Y) \subset \mathbb{PT},$$

as long as the fibres of $\mu|_Y$ are connected and simply connected.

We have glossed over the details of how to relate the various sheaves on X, Y, and Z.

4 References

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