Cohomology of Elementary States in Twistor Conformal Field Theory

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Abstract

In Twistor Conformal Field Theory the Riemann surfaces and holomorphic functions of two-dimensional conformal field theory are replaced by "flat" twistor spaces (arising from conformally-flat four-manifolds) and elements of the holomorphic first cohomology. The analogue of a Laurent Series is the expansion of a cohomology element in "elementary states" and we calculate the dimension of the space of these states for twistor spaces of compact hyperbolic manifolds. Our method follows the strategy used in the classical problem of calculating the number of meromorphic functions with prescribed poles on a Riemann surface. We express the problem globally (in terms of the cohomology of a blown-up twistor space), calculate the holomorphic Euler characteristic of this blown-up space, and then use some vanishing theorems to isolate the first cohomology term.

1 Introduction

In two-dimensional conformal field theory the interactions of quantum field theory are determined by the intrinsic properties of complex manifolds instead of by equations or action principles. There one has Riemann surfaces X with boundary ∂X the union of p positively oriented S^1's and q negatively oriented S^1's, and a "Hilbert space representation" ρ yielding

$$\rho(S^1) = \mathcal{H}$$

and

$$\rho(X) : \mathcal{H}^{\otimes p} \otimes \mathcal{H}^{\otimes q} \to \mathbf{C}$$

(among other things, such as the contraction operation) for a given Hilbert space $\mathcal{H}$.

There are two possible generalisations of this [20], depending on whether we are thinking of the conformal or complex structure of the surfaces X : (i) conformally-flat Riemannian four-manifolds M with ∂M the disjoint union of round S^3's, or (ii) "flat" twistor spaces Z with ∂Z C-R equivalent to the disjoint union of **PN**s.

Definition 1.1 *Z is flat if and only if it is the twistor space of a conformally flat four-manifold (which is the case if and only if each point in Z has a neighbourhood biholomorphic to the neighbourhood of a line in $\mathbf{CP}^3$).*

We can think of (i) above as the real case of (ii), namely when Z is fibred by a subfamily M of lines with a Riemannian self-dual conformal structure. So we focus on (ii), which was first suggested in [11] (see also [17]). In the two-dimensional conformal field theory the construction of ρ starts with the space $\mathcal{H}$ of L^2 spinors on S^1 with the usual polarisation $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ (given by continuation into the northern or southern hemisphere). Then $\rho(X)$ depends on the properties of the projections

$$\mathcal{O}(X) \subset C^\infty(\partial X) \to \mathcal{H}_+, \mathcal{H}_-.$$

In the twistor case we have already got a space of fields associated with each boundary component: $H^1(\mathbf{PN}; \mathcal{O}(-2s-2))$ is isomorphic to the space of smooth spin s fields on compactified Minkowski space [1, 2]. We also have a polarisation:

$$H^1(\mathbf{PN}; \mathcal{O}(-2s-2)) = H^1(\overline{\mathbf{P}}^+; \mathcal{O}(-2s-2)) \oplus H^1(\overline{\mathbf{P}}^-; \mathcal{O}(-2s-2))$$

(which one can obtain using the Mayer-Vietoris sequence on the cover $\overline{\mathbf{P}}^+, \overline{\mathbf{P}}^-$ of $\mathbf{CP^3}$). We thus have the Hilbert space of states $\rho(\mathbf{PN}) = H^1(\mathbf{PN}; \mathcal{O}(-2s-2))$ and a polarisation. (Note how neatly the latter arises.) How would $\rho(Z)$ provide us with a functional on the various incoming and outgoing fields? We follow the example of the two-dimensional conformal field theory and study the map

$$i^* : H^1(Z; \mathcal{O}(-2s-2)) \to H^1(\partial Z; \mathcal{O}(-2s-2)).$$

It is shown in [20] that i^* is injective and that its projections onto the positive and negative frequency parts are of the required type. This is done by using the analogue of the expansion in Laurent series of a function on a Riemann surface. This analogue is the expansion in elementary states.

$$\omega \in H^1(\mathbf{CP^3} - L; \mathcal{O}(m)) \text{ implies that } \omega = \Sigma_{r,s>0} \frac{A_{r,s}(z_2, z_3)}{(z_0)^r (z_1)^s},$$

where $A_{r,s}$ is a homogenous polynomial of degree $r+s+m$.

An elementary state is, by definition, a finite linear combination of the terms in ω; we say it is "based on L", and because

$$H^1(\mathbf{CP^3} - L; \mathcal{O}(m)) \to H^1(\mathbf{P}^+; \mathcal{O}(m))$$

is injective we can regard it as being in the latter group. In fact the elementary states are dense in $H^1(\mathbf{PN}; \mathcal{O}(m))$, those based on a line in $\mathbf{P}^-$ being dense in $H^1(\overline{\mathbf{P}}^+; \mathcal{O}(m))$ and vice versa [4]. They have long been used as a powerful calculus in twistor theory, especially in the study of twistor diagrams [12, 16].

In this paper we calculate the dimension of the space of elementary states based on a line with a prescribed order of "codimension-two" singularity there. (A summary of the result and the methods used has already appeared [14].) The line, L, is not in $\mathbf{CP^3}$ but is sitting in one of our flat twistor spaces Z.

The corresponding calculation for L in $\mathbf{CP^3}$ was done in [3], using the following characterisation of the elementary states.

For the moment let $\tilde{Z}$ be $\mathbf{CP^3}$ blown up along the line L, and let $\mathcal{O}(a,b)$ be the restriction to $\tilde{Z}$ of $\mathcal{O}(a)\times\mathcal{O}(b)$ on $\mathbf{CP}^3\times\mathbf{CP}^1$. Then there is a restriction map

$$\rho : H^1(\tilde{Z};\mathcal{O}(a,b)) \to H^1(\mathbf{CP^3}-L;\mathcal{O}(a+b))$$

which is injective, and whose image is precisely the elementary states based on L, so long as $a\geq 0$ and $b\leq -2$. These elementary states have a singularity of order up to $-b-1$. They are

$$\Sigma_{r,s>0;\ r+s\leq -b}\frac{A_{r,s}(z_2,z_3)}{(z_0)^r(z_1)^s}$$

(for L given by $z_0=z_1=0$). It is also true for a general compact flat twistor space Z that

$$\rho : H^1(\tilde{Z};\mathcal{O}(a,b)) \to H^1(Z-L;\mathcal{O}(a+b))$$

is injective, and we adopt as a working definition of elementary states (justified in the next section) the elements of $H^1(\tilde{Z};\mathcal{O}(a,b))$.

There is an important alternative motivation for studying these elementary states. A classical problem on compact Riemann surfaces is to ask how many linearly independent meromorphic functions there are on X which have poles of order at most n_i at P_i and no others. The solution is in three parts. (i) Let D be the divisor Σn_iP_i and let $[D]$ be its line bundle. The problem is now to find $\dim H^0(X;[D])$. (ii) Use the Riemann-Roch theorem:

$$\dim H^0 - \dim H^1 = \deg D + 1 - g.$$

(iii) Use the Kodaira vanishing theorem: if $\deg D > 2g-2$ then $H^1(X;[D])=0$. We follow a similar strategy in this paper.

(i) We convert our problem into a global one by showing, in section 2, that the space $H^1(\tilde{Z};\mathcal{O}(a,b))$ does indeed characterise elementary states, so that the above definition is the correct one. We will then need to calculate $\dim H^1(\tilde{Z};\mathcal{O}(a,b))$.

(ii) An earlier version of this work [13] used the Hirzebruch-Riemann-Roch Theorem to calculate $\chi(\tilde{Z};\mathcal{O}(a,b))$ in terms of Chern classes on $\tilde{Z}$, and then a theorem of Porteous [9] to relate these to characteristic classes on Z. There is a simpler method for calculating χ, however, which was pointed out to us by Claude LeBrun, and which we use in section 3.

(iii) Given χ we know the alternating sum of the dimensions of the cohomology groups on $\tilde{Z}$, and the last step is to show that $H^3=H^0=0$ and to calculate $\dim H^2$. This is done in section 4, using a vanishing theorem proved in [15].

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2 Codimension-two Poles on Flat Twistor Spaces

The extension of the notion of a codimension-two pole from $\mathbf{CP^3}$ to a general flat twistor space Z, will require a line bundle on $\tilde{Z}$ which coincides with $\mathcal{O}(a,b)$ when $\tilde{Z}$ is $\mathbf{CP^3}$ blown-up along a single projective line. We shall define this line bundle for a general twistor space which need not be flat (but must if m is odd satisfy a topological condition – the vanishing of the Stiefel-Whitney class in $H^2(Z,\mathbf{Z}_2)$ – in order that $\mathcal{O}(\kappa^{-\frac{m}{4}})$ is well defined). We give the definition in terms of sheaves.

Definition 2.1 *Let Z be a compact twistor space and let $L_1,\dots,L_s$ be pairwise non-intersecting lines in Z. Let $\tilde{Z}$ be the manifold Z blown-up along $L_1\cup\ldots\cup L_s$, and let the exceptional divisor be $E_1\cup\ldots\cup E_s$. Let $I_1,\dots,I_s$ be the ideal sheaves of $E_1,\dots,E_s$ respectively. Let m be any integer and let (a_i,b_i), for $i=1,\dots,s$, be pairs of integers with $a_i+b_i=m$. Then we define the sheaf $\mathcal{O}(a_1,\dots,a_s;b_1,\dots,b_s)$ by*

$$\mathcal{O}(a_1,\dots,a_s;b_1,\dots,b_s)=I_1^{\otimes b_1}\otimes\ldots\otimes I_s^{\otimes b_s}\otimes\varpi^*\mathcal{O}(\kappa^{-\frac{m}{4}}) \tag{1}$$

where κ is the canonical bundle of Z and ϖ is the blowing-down map from $\tilde{Z}$ to Z.

We shall demonstrate our assertion concerning the nature of this bundle when Z is $\mathbf{CP^3}$ blown-up along a single line. We determine the line bundle $[E]$ of the divisor E and from this we shall get the ideal sheaf of E. Let $\mathbf{CP^3}$ have the standard homogeneous coordinates $[z_0,z_1,z_2,z_3]$ and let the line L be given by $z_2=z_3=0$. The blow-up of $\mathbf{CP^3}$ along L is then the subvariety of $\mathbf{CP^3}\times\mathbf{CP^1}$ given by $z_2w_3-z_3w_2=0$, where $[w_2,w_3]$ are homogeneous coordinates for (the vertical) $\mathbf{CP^1}$, [7]. Call this subvariety $\tilde{\mathbf{CP^3}}$. Now construct an open cover of $\tilde{\mathbf{CP^3}}$ from the restriction of the following cover of $\mathbf{CP^3}\times\mathbf{CP^1}$. Let

$$\begin{array}{rcll} U_i &=& \{\underline{z}:z_i\neq 0\} & for\quad i=0,1,2,3\\ V_j &=& \{\underline{w}:w_j\neq 0\} & for\quad j=2,3. \end{array} \tag{2}$$

Then $\{U_i\times V_j : i,j\}$ covers $\mathbf{CP^3}\times\mathbf{CP^1}$. Since the line to be blown-up is $z_2=z_3=0$, neither U_2 nor U_3 will meet L so that a cover of the exceptional divisor is provided by the restrictions to $\tilde{\mathbf{CP^3}}$ of the sets

$$\begin{array}{ll} \mathcal{U}_0=U_0\times V_2 & \mathcal{U}_1=U_0\times V_3\\ \mathcal{U}_2=U_1\times V_2 & \mathcal{U}_3=U_1\times V_3 \end{array} \tag{3}$$

The bundle $[E]$ is determined by the transition function $g_{ij}=f_i/f_j$, where f_i is a defining function for the divisor in $\mathcal{U}_i$. Local coordinates for the blow-up are

$$\begin{array}{lll} \text{in} & \mathcal{U}_0 & \{(\frac{z_1}{z_0},\frac{z_2}{z_0},\frac{w_3}{w_2}):z_0w_2\neq 0\}\\ \text{in} & \mathcal{U}_3 & \{(\frac{z_0}{z_1},\frac{z_3}{z_1},\frac{w_2}{w_3}):z_1w_3\neq 0\} \end{array} \tag{4}$$

The exceptional divisor is given by $z_2/z_0=0$ in $\mathcal{U}_0$ and by $z_2/z_1=0$ in $\mathcal{U}_3$. Taking $u=z_1/z_0, v=z_2/z_0, w=w_3/w_2$ in $\mathcal{U}_0\cap\mathcal{U}_3$, the defining function in the former open set is v whilst in the latter it is vw/u. The transition function for this intersection of open sets is thus $g_{30}=wu^{-1}$. This is the transition function for the bundle $\mathcal{O}(1,-1)$ (where we are using now the definition of [3] for $\mathcal{O}(a,b)$, with $a=1, b=-1$). The ideal sheaf of E is thus $\mathcal{O}(-1,1)$ and hence our definition (1) coincides with that of [3] in this case. To ensure that elements of $H^1(\mathbf{C\tilde{P}^3}, \mathcal{O}(a,b))$ are good representatives for elementary states Eastwood and Hughston restricted the values of the parameters a,b to $a \geq 0$ and $b \leq -2$. In our case we make a similar restriction, i.e. when considering codimension-two poles we shall restrict the parameter values to $a_i \geq 0$ and $b_i \leq -2$ for all $i = 1 \ldots s$. We make this clear by proposing the following definition.

Definition 2.2 *Let $\tilde{Z}$ be the compact twistor space Z blown-up along the projective lines $L_1, \ldots, L_s$ as above and let the bundle $\mathcal{O}(a_1, \ldots, a_s; b_1, \ldots, b_s)$ be the bundle defined in (1). We shall say that an element η of $H^1(Z-L, \mathcal{O}(m))$ has a codimension-2 pole of order at most l_i on L_i , for $l_i \geq 0$, and $i = 1, \ldots, s$, if there exist $a_i \geq 0, b_i \leq -2$ with $a_i + b_i = m$ and $l_i = -b_i - 1$ such that η is in the image of the restriction map $H^1(\tilde{Z}, \mathcal{O}(a_1, \ldots, a_s; b_1, \ldots, b_s) \to H^1(Z-L, \mathcal{O}(m))$.*

To give some substance to this definition we shall prove that the above restriction map is injective.

Proposition 2.3 *Let Z be a compact flat twistor space and $\tilde{Z}$ be its blow-up along $L_1, \ldots, L_s$ as above. Let m be any integer and let a_i, b_i be pairs of integers with $a_i \geq 0, b_i \leq -2$ and $a_i + b_i = m$, for $i = 1, \ldots, s$. Then the restriction map*

$$H^1(\tilde{Z}, \mathcal{O}(a_1, \ldots, a_s; b_1, \ldots, b_s)) \to H^1(Z-L, \mathcal{O}(m))$$

is injective.

Proof We shall confine attention to the case of a single line L and the general case will follow. Since Z is a flat twistor space there is an open neighbourhood of L which is biholomorphic to $\mathbf{P}^+$ in $\mathbf{CP^3}$. Let the blow-up of $\mathbf{P}^+$ be $\tilde{\mathbf{P}}^+$. The local cohomology exact sequence [18] is then

$$\to H^1_E(\tilde{Z}, \mathcal{O}(a,b)) \to H^1(\tilde{Z}, \mathcal{O}(a,b)) \to H^1(Z-L, \mathcal{O}(m)) \to \tag{5}$$

with the second map being restriction. We must show that the first term in the sequence above is zero. Since $E \subset \tilde{\mathbf{P}}^+ \subset \tilde{Z}$ and $E \subset \tilde{\mathbf{P}}^+ \subset \mathbf{C\tilde{P}^3}$ the excision theorem for local cohomology [18] then gives

$$H^1_E(\tilde{Z}, \mathcal{O}(a,b)) = H^1_E(\tilde{\mathbf{P}}^+, \mathcal{O}(a,b)) = H^1_E(\mathbf{C\tilde{P}^3}, \mathcal{O}(a,b)) \tag{6}$$

so that it will suffice to prove that this latter group is zero. This may be deduced from the local cohomology exact sequence for $\mathbf{C\tilde{P}^3}$

$$\begin{aligned} &\to H^0(\mathbf{C\tilde{P}^3}, \mathcal{O}(a,b)) \xrightarrow{r_0} H^0(\mathbf{CP^3} - L, \mathcal{O}(m)) \to H^1_E(\mathbf{C\tilde{P}^3}, \mathcal{O}(a,b)) \to \\ &\to H^1(\mathbf{C\tilde{P}^3}, \mathcal{O}(a,b)) \xrightarrow{r_1} H^1(\mathbf{CP^3} - L, \mathcal{O}(a,b)) \to \end{aligned} \tag{7}$$

with r_0 and r_1 the respective restriction maps. When $a \geq 0$ and $b \leq -2$ it is shown in [3] that the former is an isomorphism and the latter is injective. This proves the proposition. □

Having established the injectivity of the restriction map we now wish to determine the nature of its image. This will give further substance to our definition of codimension-two poles. We choose coverings of $\tilde{Z}$ and $Z - L$ consisting of Stein open sets and this will enable us to display the singularity structure of elements in the image of this map. We shall consider the case of a single line since this will exhibit all the relevant features. (For the definition and properties of Stein sets see [6].)

We begin by choosing a covering of Z by Stein open sets as follows. The line L is contained in a neighbourhood which is biholomorphic to (and which, for convenience, we shall refer to as) $\mathbf{P}^+$. Cover L by Stein open sets contained in $\mathbf{P}^+$ and call these sets $\{A_i\}$. Cover the rest of $Z - L$ by Stein open sets. Now think of $\mathbf{P}^+$ as a subspace of $\mathbf{CP^3}$ and let the line L be given by $z_2 = z_3 = 0$, as above. In this setting the sets $U_j = \{\underline{z} : z_j \neq 0\}$ for $j = 2, 3$ is a covering of $\mathbf{CP^3} - L$ by Stein open sets. Then $\{A_i \cap U_j\}$ is a covering of $\mathbf{P}^+ - L$ by Stein open sets. The cover of $Z - L$ formed from these sets together with the other sets covering $Z - L$, we shall call $\mathcal{V}$.

In a similar fashion we may construct a Stein open covering of $\tilde{Z}$. Begin with the original cover of Z given above, so that $A_i \cap L \neq \emptyset$ and all A_i are in $\mathbf{P}^+$. The blow-up of $\mathbf{P}^+$ along L is a subvariety of $\mathbf{P}^+ \times \mathbf{CP^1}$ so that any Stein open cover of the latter set, when restricted to $\tilde{\mathbf{P}}^+$, is a Stein open cover of $\tilde{\mathbf{P}}^+$. If the (vertical) $\mathbf{CP^1}$ has coordinates $[w_2, w_3]$ then $\tilde{\mathbf{P}}^+$ is the subvariety of $\mathbf{P}^+ \times \mathbf{CP^1}$ given by $z_2 w_3 - z_3 w_2 = 0$. The cover $V_j = \{\underline{w} : w_j \neq 0\}$ for $j = 2, 3$ is then a Stein open cover of $\mathbf{CP^1}$. The restriction to $\tilde{\mathbf{P}}^+$ of the sets $\{A_i \times V_j\}$ is then a Stein open cover of that set. These, together with the original Stein sets on $Z - L$, form a cover of $\tilde{Z}$, which we shall label $\mathcal{W}$.

Proposition 2.4 *Let $\tilde{Z}$ be the blow-up of the compact flat twistor space Z along the line L, as above, and let $\mathcal{W}$ and $\mathcal{V}$ be the covers of $\tilde{Z}$ and $Z - L$ constructed in the previous paragraph. Let $a_i \geq 0$ and $b_i \leq -2$.*

(a) Let $\{\tilde{\rho}_{\alpha\beta}\}$ be a Cech 1- cocycle for the cover $\mathcal{W}$, representing an element of $H^1(\tilde{Z}, \mathcal{O}(a, b))$ and let $\{\rho_{\alpha\beta}\}$ be its restriction to $Z - L$. Then $\{\rho_{\alpha\beta}\}$ is a Cech 1-cocycle for the cover $\mathcal{V}$ and if $\rho_{\alpha\beta}$ is defined on $(A_i \cap U_j) \cap (A_k \cap U_l)$, with $A_i \cap A_k \cap L \neq \emptyset$, then $\rho_{\alpha\beta}$ has the form

$$\frac{h_0 g_0}{(z_2)^{-b}} + \frac{h_1 g_1}{z_2^{-b-1} z_3} + \ldots + \frac{h_{-b-1} g_{-b-1}}{z_2 z_3^{-b-1}} + \frac{h_{-b} g_{-b}}{(z_3)^{-b-1}} \tag{8}$$

where the g_i are holomorphic homogeneous functions of (z_2, z_3), with homogeneity zero, defined on $U_j \cap U_l$, and the h_i are holomorphic functions homogeneous of degree a, with a holomorphic extension to $A_i \cap A_j$.

(b) Conversely if $\{\rho_{\alpha\beta}\}$ is a Cech 1-cocycle for the cover $\mathcal{V}$ representing an element of $H^1(Z - L, \mathcal{O}(m))$, and if $\{\rho_{\alpha\beta}\}$is defined on $(A_i \cap U_j) \cap (A_k \cap U_l)$, where $A_i \cap A_k \cap L \neq \emptyset$, and has the form (8), then there is a Cech 1-cocycle $\{\tilde{\rho}_{\alpha\beta}\}$ for $\mathcal{W}$, representing an element of $H^1(\tilde{Z}, \mathcal{O}(a, b))$, whose restriction to $Z - L$ is in the cohomology class of $\{\rho_{\alpha\beta}\}$.

Proof (a) Let $\{\tilde{\rho}_{\alpha\beta}\}$ be defined on the sets $(A_i \times V_j) \cap (A_k \times V_l) = (A_i \cap A_k) \times (V_j \cap V_l)$, where $A_i \cap A_k \cap L \neq \emptyset$. The 1-cocycle $\{\tilde{\rho}_{\alpha\beta}\}$ is then a sum of elements of the form $d(z)e(w) \mid_{\tilde{\mathbf{P}}^+}$, where $d(z) \in \mathcal{O}(a)(A_i \cap A_k)$ and $e(w) \in \mathcal{O}(b)(V_j \cap V_l)$. The function $e(w)$ can be written as a Laurent series of the form

$$e(w) = \frac{1}{w_2^{-b}} \sum_{r=0}^{\infty} a_r \left(\frac{w_3}{w_2}\right)^r + \frac{1}{w_3^{-b}} \sum_{s=0}^{\infty} c_s \left(\frac{w_2}{w_3}\right)^s + \frac{e_1}{w_2 w_3^{-b-1}} + \ldots + \frac{e_{-b-1}}{w_2^{-b-1} w_3} \tag{9}$$

When d(z)e(w) is restricted to $\tilde{\mathbf{P}}^+$ and away from the exceptional divisor (so that $w_2 : w_3 = z_2 : z_3$), it becomes

$$\frac{d(z)}{z_2^{-b}}\sum_{r=0}^{\infty} a_r\left(\frac{z_3}{z_2}\right)^r + \frac{d(z)}{z_3^{-b-1}}\sum_{s=0}^{\infty} c_s\left(\frac{z_2}{z_3}\right)^s + \frac{d(z)e_1}{z_2 z_3^{-b-1}} + \ldots + \frac{d(z)e_{-b-1}}{z_2^{-b-1}z_3} \tag{10}$$

and this is the form described in (8).

(b) Conversely, suppose that $\{\rho_{\alpha\beta}\}$ is a Cech 1-cocycle for the covering $\mathcal{V}$ with the stated form, i.e. $\rho_{\alpha\beta}$ is defined on $(A_i \cap U_j) \cap (A_k \cap U_l)$ with $A_i \cap A_j \cap L \neq \emptyset$, and

$$\rho_{\alpha\beta} = \sum_{t=0}^{-b} \frac{h_t g_t}{z_2^{-b-t} z_3^t} \tag{11}$$

where h_t is holomorphic and homogeneous of degree a on the whole of $A_i \cap A_k$, and g_t is a holomorphic function of (z_2, z_3) defined on $U_j \cap U_l$ and homogeneous of degree zero. Now choose $\tilde{\rho}_{\alpha\beta}$ on the set $(A_i \times V_j) \cap (A_k \times V_l) = (A_i \cap A_k) \times (V_j \cap V_l)$ as follows. The g_t of (11) are holomorphic functions of (z_2, z_3) only, with homogeneity zero, so that $g_t(\underline{w})/(w_2^{-b-t}w_3^t) \in \mathcal{O}(b)(V_j \cap V_l)$ (since g_t is defined on $V_j \cap V_l$). The $h_t(z)$ are holomorphic on $A_i \cap A_k$ and homogeneous of degree a, so that if we take

$$\tilde{\rho}_{\alpha\beta} = \left(\sum_{t=0}^{-b} h_t(z)\frac{g_t(w)}{w_2^{-b-t}w_3^t}\right)|_{\tilde{\mathbf{P}}^+} \tag{12}$$

then $\tilde{\rho}_{\alpha\beta}$, when restricted away from the exceptional divisor to $Z - L$, is precisely $\rho_{\alpha\beta}$. It remains only to show that $\{\tilde{\rho}_{\alpha\beta}\}$ satisfies the cocycle condition. If $\rho_{\alpha\beta} - \rho_{\alpha\gamma} + \rho_{\beta\gamma} = 0$ and the common domain of these three elements is $(A_i \cap U_j) \cap (A_k \cap U_l) \cap (A_m \cap U_n) \neq \emptyset$ with $A_i \cap A_k \cap A_m \cap L \neq \emptyset$, then $\tilde{\rho}_{\alpha\beta} - \tilde{\rho}_{\alpha\gamma} + \tilde{\rho}_{\beta\gamma}$ is a holomorphic function defined on the set $(A_i \cap A_k \cap A_m) \times (V_j \cap V_l \cap V_n)$. Its restriction to $\tilde{\mathbf{P}}^+$ away from the exceptional divisor is $\rho_{\alpha\beta} - \rho_{\alpha\gamma} + \rho_{\beta\gamma}$ and this vanishes on its (open) domain. Since $\tilde{\rho}_{\alpha\beta} - \tilde{\rho}_{\alpha\gamma} + \tilde{\rho}_{\beta\gamma}$ is holomorphic and zero on an open set, it is identically zero. □

Remarks (a) The form of the singularity given by (8) has a strong resemblance to that of an elementary state, though it has the extra terms $h_0 g_0/z_2^{-b}$ and $h_{-b}g_{-b}/z_3^{-b}$. In the case of elementary states these represent coboundary terms. It seems likely that this is also true in general, though we have not been able to prove this.

(b) We now have an identification of the elements of $H^1(\tilde{Z}, \mathcal{O}(a_1, \ldots, a_s; b_1, \ldots, b_s))$ with those elements of $H^1(Z - L, \mathcal{O}(m))$ having codimension-2 poles on each L_i of prescribed maximum order.

3 Calculation of the Euler Characteristic

The calculation of the holomorphic Euler characteristic of the bundle $\mathcal{O}(a_1, \ldots, a_s; b_1, \ldots, b_s)$ on $\tilde{Z}$ is a relatively simple matter, and can be found in terms of $\chi(Z, \mathcal{O}(m))$. Recall that if S is a submanifold of codimension-1 of a compact complex manifold X, and if F is a holomorphic line bundle on X, then there is a short exact sequence

$$0 \to \mathcal{O}(F \otimes [S]^*) \to \mathcal{O}(F) \xrightarrow{r_S} \mathcal{O}(F_S) \to 0 \tag{13}$$

where $[S]^*$ is the dual of the line bundle of the divisor S, F_S is the restriction of F to S, with restriction map r_S [19]. In our case, if we first consider only the divisor E_1 then we have the short exact exact sheaf sequence

$$0 \to \mathcal{O}(a_1 - 1, \ldots, a_s; b_1 + 1, \ldots, b_s) \to \mathcal{O}(a_1, \ldots, a_s; b_1, \ldots, b_s) \to \mathcal{O}_{E_1}(a_1; b_1) \to 0 \tag{14}$$

where the last term is $\mathcal{O}(a_1, \ldots, a_s; b_1, \ldots b_s)$ restricted to E_1. Thus we obtain

$$\begin{aligned} \chi(\tilde{Z}, \mathcal{O}(a_1, \ldots, a_s; b_1, \ldots b_s)) - \chi(\tilde{Z}, \mathcal{O}(a_1 - 1, \ldots, a_s; b_1 + 1, \ldots b_s)) &= \chi(E_1, \mathcal{O}(a_1, b_1)) \\ &= (1 + a_1)(1 + b_1) \end{aligned} \tag{15}$$

since each E_i is a $\mathbf{CP^1} \times \mathbf{CP^1}$. A simple induction argument for this and the other E_j quickly gives

$$\begin{aligned} \chi(\tilde{Z}, \mathcal{O}(a_1, \ldots, a_s; b_1, \ldots, b_s)) &= \chi(Z, \mathcal{O}(m)) - \frac{1}{6}\sum_{j=1}^{s} b_j(b_j + 1)(3m - 2b_j + 5) \\ &= \frac{1}{12}(m + 1)(m + 2)(m + 3)\chi \\ &\quad -\frac{1}{8}(m + 2)[(m + 1)(m + 3) - 1]\tau \\ &\quad -\frac{1}{6}\sum_{i=1}^{s} b_i(b_i + 1)(3m - 2b_i + 5) \end{aligned} \tag{16}$$

where χ is the Euler characteristic of the compact Riemannian 4-manifold X, and τ is its signature. This follows from the fact that $\chi(\tilde{Z}, \mathcal{O}(m, \ldots, m; 0, \ldots, 0)) = \chi(Z, \mathcal{O}(m))$ and the value of this has already been calculated. (See [5] for instance, or it can be calculated from the Chern classes for Z in [10].)

4 Analytic Cohomology on Flat and Blown-Up Twistor Space

In order to find the dimension of the H^1 term in the alternating sum which makes up the holomorphic Euler characteristic, it is necessary to determine the dimensions of the other terms. Instead of proceeding directly we shall show that, in some cases, it will be sufficient to have this information for cohomology groups on the flat twistor space. This will allow the use of vanishing theorems for compact flat twistor spaces, some of which are well known. The awkward term term to deal with is the H^2 term. We first find its Serre dual when the exceptional divisor E is a single irreducible quadric and the general case will again be obvious.

The Serre dual [21] of $H^2(\tilde{Z}, \mathcal{O}(a, b))$ is $H^1(\tilde{Z}, \kappa_{\tilde{Z}} \otimes \mathcal{O}(a, b)^*)$ where $\kappa_{\tilde{Z}}$ is the canonical bundle of $\tilde{Z}$ and $\mathcal{O}(a, b)^*$ is the dual bundle. This latter is plainly $\mathcal{O}(-a, -b)$. The canonical bundle can be found using the fact [8] that, for a blown-up manifold, $\kappa_{\tilde{Z}} = \varpi^*(\kappa_Z) \otimes [E]$ where ϖ is the blow-down map. The former bundle is $\mathcal{O}(-4, 0)$ and the latter is $\mathcal{O}(1, -1)$, so that $\kappa_{\tilde{Z}} = \mathcal{O}(-3, -1)$. In the general case, one now has that

$$H^2(\tilde{Z}, \mathcal{O}(a_1, \ldots, a_s; b_1, \ldots b_s)) \cong H^1(\tilde{Z}, \mathcal{O}(c_1, \ldots, c_s; d_1, \ldots d_s)) \tag{17}$$

where $c_i = -a_i - 3$ and $d_i = -b_i - 1$ for $i = 1, \ldots, s$. For codimension-2 poles the further restrictions $a_i \geq 0$ and $b_i \leq -2$ must be imposed, so that, in this case, $c_i \leq -3$ and $d_i \geq 1$. Then for each i, one has $c_i + d_i = n$, with $n = -m - 4$.

Our next task is to establish some of the properties of the cohomology groups $H^1(\tilde{\mathbf{P}}^+, \mathcal{O}(c,d))$ and $H^1(\mathbf{P}^+, \mathcal{O}(c+d))$ which will be required later in this section. Here we are again considering the case of a single line L in a neighbourhood $\mathbf{P}^+$, so that $\tilde{\mathbf{P}}^+$ is $\mathbf{P}^+$ blown-up along a single line. We shall show that when $c \leq -3$ and $d \geq 1$, there is a monomorphism from $H^1(\tilde{\mathbf{P}}^+, \mathcal{O}(c,d))$ into $H^1(\mathbf{P}^+, \mathcal{O}(c+d))$, and this quickly extends to the general case. The existence of this monomorphism is established in the next three lemmas.

Lemma 4.1 *If $c \leq -3$ and $d \geq 1$ then every element of $H^1(\tilde{\mathbf{P}}^+, \mathcal{O}(c,d))$ can be represented as the restriction to $\tilde{\mathbf{P}}^+$ of a sum of terms of the form $f(z)g(w)$ for some $f \in H^1(\mathbf{P}^+, \mathcal{O}(c))$ and $g \in H^0(\mathbf{CP^1}, \mathcal{O}(d))$.*

Proof The method is a simple adaptation of [3]. The bundle $\mathcal{O}(c,d)$ can be defined by the short exact sheaf sequence

$$0 \to \mathcal{O}_{\mathbf{P}^+ \times \mathbf{CP^1}}(c-1, d-1) \xrightarrow{\sigma} \mathcal{O}_{\mathbf{P}^+ \times \mathbf{CP^1}}(c,d) \xrightarrow{\rho} \mathcal{O}_{\tilde{\mathbf{P}}^+}(c,d) \to 0 \tag{18}$$

where the first map is multiplication by the ideal sheaf of $\tilde{\mathbf{P}}^+$, and the second is restriction to this subvariety. This leads to a long exact sequence in cohomology which, for $c \leq -3$ and $d \geq 1$, yields

$$\begin{aligned} 0 \to H^0(\tilde{\mathbf{P}}^+, \mathcal{O}(c,d)) \to H^1(\mathbf{P}^+ \times \mathbf{CP^1}, \mathcal{O}(c-1, d-1)) \to \\ \to H^1(\mathbf{P}^+ \times \mathbf{CP^1}, \mathcal{O}(c,d)) \to H^1(\tilde{\mathbf{P}}^+, \mathcal{O}(c,d)) \to 0 \end{aligned} \tag{19}$$

the penultimate map being restriction. The zeros in (19) are a consequence of the vanishing of the respective cohomology groups, which can be verified by using the Künneth formula for sheaf cohomology. This may also be used to prove that if $c \leq -3$ and $d \geq 1$ then $H^1(\mathbf{P}^+ \times \mathbf{CP^1}, \mathcal{O}(c,d)) = H^1(\mathbf{P}^+, \mathcal{O}(c)) \otimes H^0(\mathbf{CP^1}, \mathcal{O}(d))$ and the lemma follows. □

Lemma 4.2 *Let $c \leq -3$ and $d \geq 1$. If $f_1 \in H^1(\tilde{\mathbf{P}}^+, \mathcal{O}(c,d))$ then there exists $f_2 \in H^1(\mathbf{P}^+, \mathcal{O}(c+d))$ such that $f_1 \big|_{\tilde{\mathbf{P}}^+ - E} = f_2 |_{\mathbf{P}^+ - L}$ where we have identified $\tilde{\mathbf{P}}^+ - E$ with $\mathbf{P}^+ - L$.*

Proof Suppose $f_1 \in H^1(\tilde{\mathbf{P}}^+, \mathcal{O}(c,d))$. Then by (4.1) we can find $f(z) \in H^1(\mathbf{P}^+, \mathcal{O}(c))$ and $g(w) \in H^0(\mathbf{CP^1}, \mathcal{O}(d))$ such that $f_1 = f(z)g(w) |_{\tilde{\mathbf{P}}^+}$. Taking $[z_0, z_1, z_2, z_3]$ as coordinates on $\mathbf{P}^+$ and $[w_2, w_3]$ as coordinates on $\mathbf{CP^1}$, then in $\tilde{\mathbf{P}}^+$, and away from the exceptional divisor, we have $w_2 : w_3 = z_2 : z_3$. Thus f_1, when restricted away from the exceptional divisor, becomes $f(z)g(z) \big|_{\tilde{\mathbf{P}}^+ - L}$, since we may identify $g(z)$ with an element of $H^0(\mathbf{P}^+, \mathcal{O}(d))$. The result is now clear. □

Lemma 4.3 *The restriction maps s_1, s_2*

$$\begin{aligned} s_1 : \quad & H^1(\tilde{\mathbf{P}}^+, \mathcal{O}(c,d)) \to H^1(\tilde{\mathbf{P}}^+ - E, \mathcal{O}(c+d)) \\ s_2 : \quad & H^1(\mathbf{P}^+, \mathcal{O}(c+d)) \to H^1(\mathbf{P}^+ - L, \mathcal{O}(c+d)) \end{aligned}$$

are both monomorphisms.

Proof (a)We shall deal with s_1 first. The local cohomology exact sequence for $\tilde{\mathbf{P}}^+$ is

$$\begin{aligned} &\to H^0(\tilde{\mathbf{P}}^+, \mathcal{O}(c,d)) \to H^0(\mathbf{P}^+ - L, \mathcal{O}(c+d)) \to \\ \to H^1_E(\tilde{\mathbf{P}}^+, \mathcal{O}(c,d)) &\xrightarrow{\gamma} H^1(\tilde{\mathbf{P}}^+, \mathcal{O}(c,d)) \to H^1(\mathbf{P}^+ - L, \mathcal{O}(c+d)) \to \end{aligned} \tag{20}$$

and we shall show that γ is the zero map.

First note that the restriction map of $H^0(\tilde{\mathbf{P}}^+, \mathcal{O}(c,d))$ to $H^0(\mathbf{P}^+ - L, \mathcal{O}(c+d))$ is injective since any element of the latter is, locally, a holomorphic function, and if its restriction to the open subset $\mathbf{P}^+ - L$ is zero, then it is identically zero. Any element of $H^0(\tilde{\mathbf{P}}^+, \mathcal{O}(c,d))$, away from the exceptional divisor, is then either a homogeneous polynomial, or zero. Both cases are well defined globally on $\mathbf{C}\tilde{\mathbf{P}}^\mathbf{3} - L$, so that $H^0(\tilde{\mathbf{P}}^+, \mathcal{O}(c,d))$ is isomorphic to $H^0(\mathbf{C}\tilde{\mathbf{P}}^\mathbf{3}, \mathcal{O}(c,d))$.

The bundle $\mathcal{O}(c,d)$ can also be defined on $\mathbf{C}\tilde{\mathbf{P}}^\mathbf{3}$ in the same way as on $\tilde{\mathbf{P}}^+$, i.e. by the short exact sequence

$$0 \to \mathcal{O}_{\mathbf{CP^3}\times\mathbf{CP^1}}(c-1, d-1) \to \mathcal{O}_{\mathbf{CP^3}\times\mathbf{CP^1}}(c,d) \to \mathcal{O}_{\mathbf{C}\tilde{\mathbf{P}}^\mathbf{3}}(c,d) \to 0 \tag{21}$$

and this gives rise to the long exact sequence in cohomology

$$\begin{aligned} &\to H^0(\mathbf{CP^3}\times\mathbf{CP^1}, \mathcal{O}(c,d)) \to H^0(\mathbf{C}\tilde{\mathbf{P}}^\mathbf{3}, \mathcal{O}(c,d)) \to H^1(\mathbf{CP^3}\times\mathbf{CP^1}, \mathcal{O}(c-1,d-1)) \to \\ &\to H^1(\mathbf{CP^3}\times\mathbf{CP^1}, \mathcal{O}(c,d)) \to H^1(\mathbf{C}\tilde{\mathbf{P}}^\mathbf{3}, \mathcal{O}(c,d)) \to H^2(\mathbf{CP^3}\times\mathbf{CP^1}, \mathcal{O}(c-1,d-1)) \to \end{aligned}$$

Using the Künneth formula with $c \leq -3$ and $d \geq 1$, one can easily show that

$$H^0(\mathbf{C}\tilde{\mathbf{P}}^\mathbf{3}, \mathcal{O}(c,d)) = H^1(\mathbf{C}\tilde{\mathbf{P}}^\mathbf{3}, \mathcal{O}(c,d)) = 0 \tag{22}$$

Thus (20) becomes

$$0 \to H^0(\mathbf{P}^+ - L, \mathcal{O}(c+d)) \to H^1_E(\tilde{\mathbf{P}}^+, \mathcal{O}(c,d)) \xrightarrow{\gamma} H^0(\tilde{\mathbf{P}}^+, \mathcal{O}(c,d)) \to H^1(\mathbf{P}^+ - \mathbf{L}, \mathcal{O}(c+d)) \tag{23}$$

Since $E \subseteq \tilde{\mathbf{P}}^+ \subseteq \mathbf{C}\tilde{\mathbf{P}}^\mathbf{3}$, and since E is closed and $\tilde{\mathbf{P}}^+$ is open in $\mathbf{C}\tilde{\mathbf{P}}^\mathbf{3}$, we have $H^1_E(\mathbf{C}\tilde{\mathbf{P}}^\mathbf{3}, \mathcal{O}(c,d)) \cong H^1_E(\tilde{\mathbf{P}}^+, \mathcal{O}(c,d))$ by the excision theorem for local cohomology. The latter term may now be found from the local cohomology sequence

$$H^0(\mathbf{C}\tilde{\mathbf{P}}^\mathbf{3}, \mathcal{O}(c,d)) \to H^0(\mathbf{CP^3} - L, \mathcal{O}(c+d)) \to H^1_E(\mathbf{C}\tilde{\mathbf{P}}^\mathbf{3}, \mathcal{O}(c,d)) \to H^1(\mathbf{C}\tilde{\mathbf{P}}^\mathbf{3}, \mathcal{O}(c,d)) \tag{24}$$

Using (22) we deduce that $H^0(\mathbf{CP^3} - L, \mathcal{O}(c+d)) \cong H^1_E(\mathbf{C}\tilde{\mathbf{P}}^\mathbf{3}, \mathcal{O}(c,d))$. Since $H^0(\mathbf{CP^3} - L, \mathcal{O}(c+d)) \cong H^0(\mathbf{CP^3}, \mathcal{O}(c+d)) \cong H^0(\mathbf{P}^+, \mathcal{O}(c+d)) \cong H^0(\mathbf{P}^+ - L, \mathcal{O}(c+d))$, it is apparent that γ is indeed the zero map.

(b) For s_2 we have the local cohomology sequence

$$\to H^1_L(\mathbf{P}^+, \mathcal{O}(n)) \to H^1(\mathbf{P}^+, \mathcal{O}(n)) \to H^1(\mathbf{P}^+ - L, \mathcal{O}(n)) \to \tag{25}$$

We use the excision theorem again to obtain $H^1_L(\mathbf{P}^+, \mathcal{O}(n)) \cong H^1_L(\mathbf{CP^3}, \mathcal{O}(n))$. This latter group can be found from the local cohomology exact sequence

$$H^0(\mathbf{CP^3}, \mathcal{O}(n)) \to H^0(\mathbf{CP^3} - L, \mathcal{O}(n)) \to H^1_L(\mathbf{CP^3}, \mathcal{O}(n)) \to H^1(\mathbf{CP^3}, \mathcal{O}(n)) \tag{26}$$

Since $H^1(\mathbf{CP^3}, \mathcal{O}(n)) = 0$ and $H^0(\mathbf{CP^3}, \mathcal{O}(n)) \to H^0(\mathbf{CP^3} - L, \mathcal{O}(n))$ is an isomorphism, $H^1_L(\mathbf{P}^+, \mathcal{O}(n)) = 0$.□

As a consequence of the last lemma we have the following.

Proposition 4.4 *For $c \leq -3$ and $d \geq 1$ there are monomorphisms*

$$r_1 : H^1(\tilde{\mathbf{P}}^+, \mathcal{O}(c,d)) \to H^1(\mathbf{P}^+, \mathcal{O}(c+d))$$
$$r_0 : H^0(\tilde{\mathbf{P}}^+, \mathcal{O}(c,d)) \to H^0(\mathbf{P}^+, \mathcal{O}(c+d))$$

Proof For the former simply take $r_1 = s_2^{-1} s_1$, where s_1, s_2 are as in lemma 4.3. For the latter it is enough to note that the restriction map $H^0(\tilde{\mathbf{P}}^+, \mathcal{O}(c,d)) \to H^0(\mathbf{P}^+ - L, \mathcal{O}(c+d))$ is injective, and the restriction map $H^0(\mathbf{P}^+, \mathcal{O}(c+d)) \to H^0(\mathbf{P}^+ - L, \mathcal{O}(c+d))$ is an isomorphism. □

The lines L_i in Z which are to be blown-up, are pairwise disjoint, so that each L_i has a neighbourhood N_i which is biholomorphic to $\mathbf{P}^+$, and these may be chosen to be pairwise disjoint. Take N to be the disjoint union of these N_i and $\tilde{N}$ to be N blown-up along each of the lines, so that $\tilde{N}$ is a disjoint union of the open sets $\tilde{N}_i$. We then have $Z = (Z - L) \cup N$ and $\tilde{Z} = (\tilde{Z} - E) \cup \tilde{N} = (Z - L) \cup \tilde{N}$ where the latter is the observation that $Z - L$ is biholomorphic to $\tilde{Z} - E$. We now have the following two Mayer-Vietoris sequences for sheaf cohomology [18].

$$\begin{array}{ccccccccc} H^0(\tilde{N}) & \overset{-h^*}{\searrow} & & & & \overset{j^*}{\nearrow} & H^1(\tilde{N}) & \overset{-h^*}{\searrow} & \\ \oplus & & H^0(N-L) & \overset{\alpha}{\longrightarrow} & H^1(\tilde{Z}) & & \oplus & & H^1(N-L) \\ H^0(Z-L) & \underset{l^*}{\nearrow} & & & & \underset{k^*}{\searrow} & H^1(Z-L) & \underset{l^*}{\nearrow} & \end{array} \tag{27}$$

$$\begin{array}{ccccccccc} H^0(N) & \overset{-m^*}{\searrow} & & & & \overset{s^*}{\nearrow} & H^1(N) & \overset{-m^*}{\searrow} & \\ \oplus & & H^0(N-L) & \overset{\beta}{\longrightarrow} & H^1(Z) & & \oplus & & H^1(N-L) \\ H^0(Z-L) & \underset{l^*}{\nearrow} & & & & \underset{t^*}{\searrow} & H^1(Z-L) & \underset{l^*}{\nearrow} & \end{array} \tag{28}$$

Here and in the proof below the sheaves, which are $\mathcal{O}(c_1, \ldots, c_s; d_1, \ldots, d_s)$ and $\mathcal{O}(n)$ on the blown-up and flat spaces respectively, have been omitted for notational convenience. The maps r_1 and r_2 of Proposition 4.4 can be extended in an obvious way to the corresponding cohomology groups for $\tilde{N}$ and N, and this fact, together with the above two Mayer-Vietoris sequences, allows a detailed analysis of the relationship between the cohomologies of $\tilde{Z}$ and Z. In particular, we are able to prove the following.

Theorem 4.5 *Let Z be a compact flat twistor space and let $L = L_1 \cup \ldots \cup L_s$, where the L_i are pairwise non-intersecting complex projective lines in Z. Let $\tilde{Z}$ be the blow-up of Z along L and let $\mathcal{O}(c_1, \ldots, c_s; d_1, \ldots, d_s)$ be the bundle on $\tilde{Z}$ described above, with $c_i \leq -3, d_i \geq 1$, and $c_i + d_i = n$ for $i = 1, \ldots, s$. If $H^0(Z, \mathcal{O}(n)) = H^1(Z, \mathcal{O}(n)) = 0$ then*

$$\dim H^1(\tilde{Z}, \mathcal{O}(c_1, \ldots, c_s; d_1, \ldots, d_s) = \begin{cases} \frac{1}{6}[s(n+1)(n+2)(n+3)] & \text{if} \quad n \geq 0 \\ 0 & \text{if} \quad n < 0 \end{cases} \tag{29}$$

Proof First note that since the restriction map $H^0(\mathbf{P}^+, \mathcal{O}(n)) \to H^0(\mathbf{P}^+ - L, \mathcal{O}(n))$ is an isomorphism, this must also be the case for the map m^* on the left of (28), so that β is the zero map. The map $s^* \oplus t^*$ is thus injective. If r_1 is the map of Proposition 4.4 and i_1 is the appropriate identity map, then $r_1 \oplus i_1$ is a monomorphism from $H^1(\tilde{N}) \oplus H^1(Z - L)$ into $H^1(N) \oplus H^1(Z - L)$. From the definition of r_1, and referring to the right hand sides of (27) and (28), one has $r_1 = (m^*)^{-1}h^*$. The kernel of the map $l^* - h^*$ now maps injectively into the kernel of $l^* - m^*$, where we are again referring to the right hand sides of both (27) and (28). There is therefore an injection of the image of the map $j^* \oplus k^*$ into $H^1(Z)$. The vanishing of $H^1(Z)$ now implies that $H^1(\tilde{Z})$ is the image of the map α , and this remains to be evaluated.

The image of α is the cokernel of $l^* - h^*$, which is $H^0(N - L)/(\mathrm{im}l^* + \mathrm{im}h^*)$. The map r_0 of Proposition 4.4 is, in this case, given by $r_0 = (m^*)^{-1}h^*$ so that $h^* = m^* r_0$. Since r_0 is a monomorphism and m^* is an isomorphism, we can identify $h^*(H^0(\tilde{N}))$ as a subgroup of $H^0(N - L)$ and $r_0(H^0(\tilde{N}))$ as a subgoup of $H^0(N)$, so that, up to isomorphism, we have

$$H^0(\tilde{N}) + \mathrm{im}l^* \subseteq H^0(N) + \mathrm{im}l^* \subseteq H^0(N - L) \tag{30}$$

Up to isomorphism therefore, the image of β is $H^0(N - L)/(H^0(N) + \mathrm{im}l^*)$ and by the second isomorphism theorem for groups, this is isomorphic to the quotient of $H^0(N - L)/(H^0(\tilde{N}) + \mathrm{im}l^*)$ by $(H^0(N) + \mathrm{im}l^*)/(H^0(\tilde{N}) + \mathrm{im}l^*)$. Since β is the zero map, and since the former group is the image of α, we have $\mathrm{im}\alpha = (H^0(N) + \mathrm{im}l^*)/(H^0(\tilde{N}) + \mathrm{im}l^*)$. From (22), since $c_i \leq -3$ and $d_i \geq 1$, it follows that $H^0(\tilde{N})$ is zero. This gives

$$\mathrm{im}\alpha = \frac{H^0(N) + \mathrm{im}l^*}{\mathrm{im}l^*} \tag{31}$$

The map l^* is injective, $N - L$ being an open subset of $Z - L$, so to complete the theorem it will suffice to show that the vanishing of $H^0(Z)$ implies the vanishing of $H^0(Z - L)$. The restriction map is clearly injective. (See the paragraph preceding (21) for a proof.) For surjectivity, note that an element of $H^0(Z-L)$ can be represented locally as a holomorphic function. Since L has codimension-2 in Z, Hartogs theorem implies that this may be extended to all of Z, so that the restriction map is surjective. We have shown that

$$\mathrm{dim}H^1(\tilde{Z}, \mathcal{O}(c_1, \ldots, c_s; d_1, \ldots, d_s) = s.\mathrm{dim}H^0(\mathbf{P}^+, \mathcal{O}(n)), \tag{32}$$

so the proof is now complete. □

We complete this section by dealing with the two end terms in the holomorphic Euler characteristic.

Theorem 4.6 *Let $\tilde{Z}, Z, \mathcal{O}(c_1, \ldots, c_s; d_1, \ldots, d_s), \mathcal{O}(n)$ be as above. Then $H^0(\tilde{Z}, \mathcal{O}(c_1, \ldots, c_s; d_1, \ldots, d_s)) = 0$ if $H^0(Z, \mathcal{O}(n)) = 0$.*

Proof This is elementary. The restriction map from $H^0(\tilde{Z}, \mathcal{O}(c_1, \ldots, c_s; d_1, \ldots, d_s))$ to $H^0(Z - L, \mathcal{O}(n))$ is injective whilst the restriction map from $H^0(Z, \mathcal{O}(n))$ to $H^0(Z - L, \mathcal{O}(n))$ is an isomorphism. □

Corollary 4.7 *With the conditions as above, if $H^3(Z, \mathcal{O}(n)) = 0$ then $H^3(\tilde{Z}, \mathcal{O}(c_1, \ldots, c_s; d_1, \ldots, d_s)) = 0$.*

Proof Noting that the proof of Theorem 4.6 does not depend upon any restrictions on the values of c_i, d_i, (apart from having $c_i + d_i = n$ so that the bundle $\mathcal{O}(c_1, \ldots, c_s; d_1, \ldots, d_s)$ is well defined), the corollary follows on taking the Serre duals of both groups, and applying the theorem.

Remarks (a) If the n of Theorem 4.5 is negative then $H^0(\mathbf{P}^+, \mathcal{O}(n))$ is zero. The $H^0(N)$ of (31) is then zero, since it is a direct sum of such groups. In this case the vanishing of $H^1(Z, \mathcal{O}(n))$ is sufficient to guarantee the vanishing of $H^1(\tilde{Z}, \mathcal{O}(c_1, \ldots, c_s; d_1, \ldots, d_s))$.

(b) As noted above, Theorem 4.6 and Corollary 4.7 do not require any restrictions on the parameters c_i, d_i apart from having $c_i+d_i = n$, which is essential for the definition of the bundle $\mathcal{O}(c_1, \ldots, c_s; d_1, \ldots, d_s)$.

First note that the restriction map of $H^0(\tilde{\mathbf{P}}^+, \mathcal{O}(c, d))$ to $H^0(\mathbf{P}^+ - L, \mathcal{O}(c + d))$ is injective since any element of the latter is, locally, a holomorphic function, and if its restriction to the open subset $\mathbf{P}^+ - L$ is zero, then it is identically zero. Any element of $H^0(\mathbf{P}^+ - L, \mathcal{O}(c + d))$ can be uniquely extended to an element of $H^0(\mathbf{P}^+, \mathcal{O}(c + d))$ and this in turn can be extended to an element of $H^0(\mathbf{CP^3}, \mathcal{O}(c+ d))$. Thus every element of $H^0(\tilde{\mathbf{P}}^+, \mathcal{O}(c, d))$ can be extended to an element of $H^0(\mathbf{C\tilde{P}^3}, \mathcal{O}(c, d))$ and these two groups are then clearly isomorphic. □

5 The Dimension of $H^1(\tilde{Z}, \mathcal{O}(a_1, \ldots, a_s; b_1, \ldots, b_s))$

Throughout this section, Z is the twistor space of a compact Riemannian self-dual 4-manifold, M. We treat separately, the cases of positive and negative scalar curvature of M.

Negative Scalar Curvature When M has negative scalar curvature, there are a number of vanishing theorems which may be utilised. In particular it is well known that $H^0(Z, \mathcal{O}(n)) = 0$ for $n > 0$ and $H^3(Z, \mathcal{O}(n)) = 0$ for $n \geq 0$. Using Serre duality one has immediately that $H^0(Z, \mathcal{O}(n)) = 0$ for $n \leq -4$ and $H^3(Z, \mathcal{O}(n)) = 0$ for $n < -4$. In addition, it is shown in [15] that if M is also Einstein, then $H^1(Z, \mathcal{O}(n)) = 0$ for $n > 0$. Noting that the conditions of theorem 4.5 require Z to be a flat twistor space, and that this corresponds to τ being zero in (16), we can make the following deduction.

Theorem 5.1 *Let M be a compact, Riemannian, conformally flat, Einstein 4-manifold, having negative scalar curvature. Let Z be its twistor space and let L and $\mathcal{O}(a_1, \ldots, a_s; b_1, \ldots, b_s)$ be as in definition 2.1, with $a_i \geq 0$ and $b_i \leq -2$. Then for $m < -4$,*

$$\begin{aligned} dim H^1(\tilde{Z}, \mathcal{O}(a_1, \ldots, a_s; b_1, \ldots, b_s)) &= \frac{1}{6}s(-m-3)(-m-2)(-m-1) \\ &\quad -\frac{1}{12}(m+1)(m+2)(m+3)\chi \\ &\quad +\frac{1}{6}\sum_{i=1}^{s} b_i(b_i+1)(3m-2b_i+5) \end{aligned}$$

where χ is the Euler characteristic of M.

Proof This is simply a matter of collecting together the relevant information. First, since $H^0(Z, \mathcal{O}(m)) = H^3(Z, \mathcal{O}(m)) = 0$ for $m < -4$, Theorem 4.6 and its corollary imply that $H^0(\tilde{Z}, \mathcal{O}(a_1, \ldots, a_s; b_1, \ldots, b_s))$ and $H^3(\tilde{Z}, \mathcal{O}(a_1, \ldots, a_s; b_1, \ldots, b_s))$ both vanish for $a_i + b_i = m < -4$.

The Serre dual of $H^2(\tilde{Z}, \mathcal{O}(a_1, \ldots, a_s; b_1, \ldots, b_s))$ is $H^1(\tilde{Z}, \mathcal{O}(c_1, \ldots, c_s; d_1, \ldots, d_s))$, with $c_i \leq -3$ and $d_i \geq 1$. Since $c_i + d_i = -m - 4 > 0$ the vanishing of $H^0(Z, \mathcal{O}(n))$ and $H^1(Z, \mathcal{O}(n))$ for $n > 0$, together with Theorem 4.5, gives the result. □

Positive Scalar Curvature When M has positive scalar curvature this method will yield only a partial answer, since there are fewer vanishing theorems for the twistor space Z. In any case, one has that $H^0(Z, \mathcal{O}(n)) = 0$ if $n < 0$ (simply by restricting to a twistor line), and in this case we also have $H^1(Z, \mathcal{O}(n)) = 0$ if $n < 0$. (Note that this does *not* require M to be Einstein). We obtain the following.

Theorem 5.2 *Let M be a compact Riemannian conformally flat 4-manifold, having positive scalar curvature. Let Z be its twistor space and let L and $\mathcal{O}(a_1, \ldots, a_s; b_1, \ldots, b_s)$ be as in definition 2.1, with $a_i \geq 0$ and $b_i \leq -2$. Then for $m \geq 0$,*

$$\begin{aligned} dim H^1(\tilde{Z}, \mathcal{O}(a_1, \ldots, a_s; b_1, \ldots, b_s)) &= dim H^0(\tilde{Z}, \mathcal{O}(a_1, \ldots, a_s; b_1, \ldots, b_s)) \\ &\quad - \frac{1}{12}(m+1)(m+2)(m+3)\chi \\ &\quad + \frac{1}{6}\sum_{i=1}^{s} b_i(b_i+1)(3m - 2b_i + 5) \end{aligned}$$

where χ is the Euler characteristic of M.

Proof From our discussions above, $H^0(Z, \mathcal{O}(-m-4)) = 0$ and $H^1(Z, \mathcal{O}(-m-4)) = 0$. From Theorem 4.5 this implies that (the Serre dual of) $H^2(\tilde{Z}, \mathcal{O}(a_1, \ldots, a_s; b_1, \ldots, b_s))$ vanishes. Since $H^3(Z, \mathcal{O}(m)) = 0$ for $m \geq 0$, we deduce that $H^3(\tilde{Z}, \mathcal{O}(a_1, \ldots, a_s; b_1, \ldots, b_s))$ is zero. This proves the result. □

Remarks (a) The conditions satisfied by the M of Theorem 5.1 are precisely those for compact hyperbolic manifolds.

(b) In the case of negative scalar curvature, $H^0(Z, \mathcal{O}(n))$ and $H^3(Z, \mathcal{O}(n))$ both vanish for $n > 0$ and for $n < -4$. If a vanishing theorem could be found for $H^1(Z, \mathcal{O}(n))$ for $n < -4$, when M is conformally flat, then this method could be used to determine the dimension of $H^1(\tilde{Z}, \mathcal{O}(a_1, \ldots, a_s; b_1, \ldots, b_s))$ when $m > 0$. Such a vanishing theorem is unlikely since it would hold when M had the additional property of being Einstein, which together with the vanishing of $H^1(Z, \mathcal{O}(n))$ when $n > 0$, would imply that the holomorphic Euler characteristic of $\mathcal{O}(n)$ on Z would be zero for $n > 0$ and for $n < -4$. This is clearly not so.

(c) In the case of positive scalar curvature the dimension of $H^1(\tilde{Z}, \mathcal{O}(a_1, \ldots, a_s; b_1, \ldots, b_s))$ has been determined in terms of $H^0(\tilde{Z}, \mathcal{O}(a_1, \ldots, a_s; b_1, \ldots, b_s))$, when $m \geq 0$, and this latter group vanishes if $H^0(Z, \mathcal{O}(m))$ vanishes. Using the Penrose transform [1, 2], one can show that an element of this group corresponds to the solution of a system of partial differential equations which is heavily overdetermined. Thus one would expect the vanishing of $H^0(Z, \mathcal{O}(m))$ for $m \geq 0$, at least generically, so that the results of Theorem 5.2 must hold in all but a few cases.

In the generic case therefore, the remarks made in (b) are equally valid when addressing the possibility of the extension of this method to the case $m < -4$, when M has positive scalar curvature.

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